Master NPAC:
An introduction to the theory of nuclear reactions

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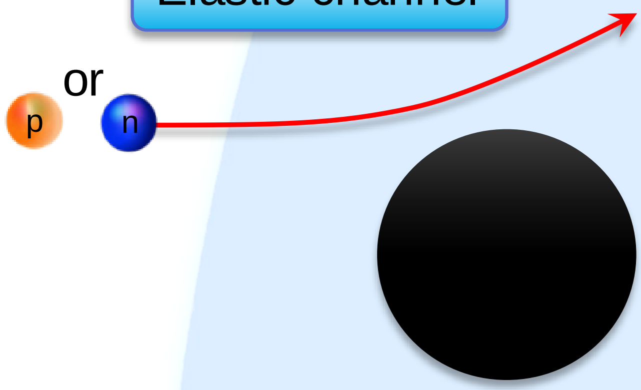
prepared with inputs of D. Lacroix

Lecture 3 : Formal theory of nuclear scattering by a general potential. Inelastic channels. Illustration with the nucleon-nucleus reaction.

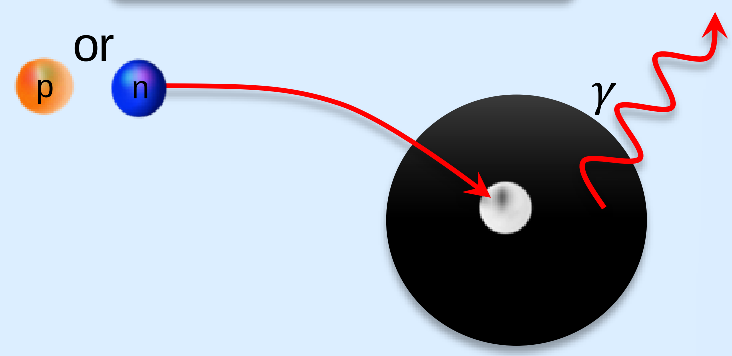
GENERALITIES ON THE NUCLEON-NUCLEUS COLLISIONS

When the incident energy permits it, two or more reaction channels are opened, e.g.

Elastic channel



Radiative capture



This may happens for astrophysical reactions

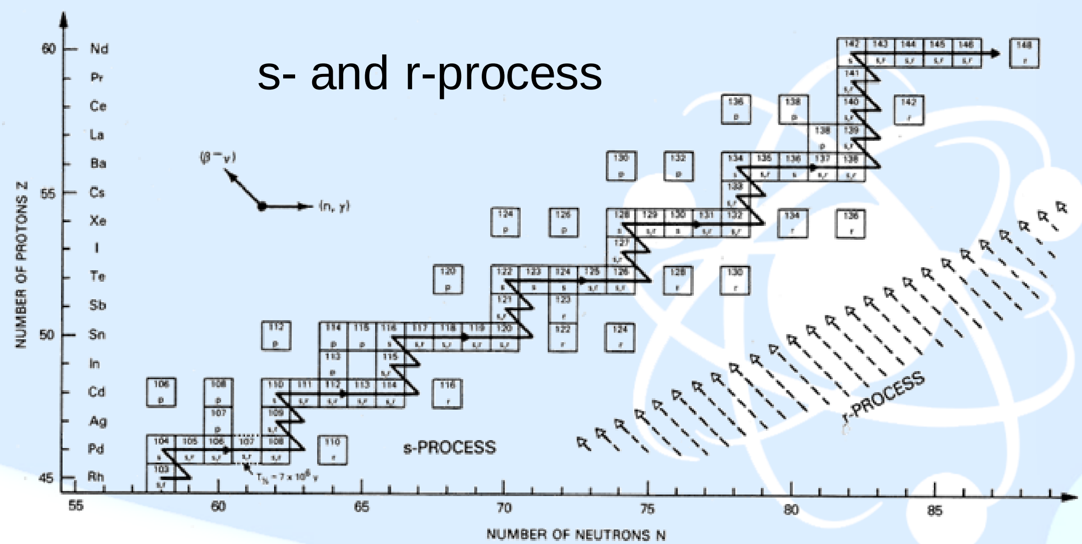
$$^{7}Be + p \rightarrow {}^{8}B + \gamma$$

$$^{13}C + p \rightarrow {}^{14}N + \gamma$$

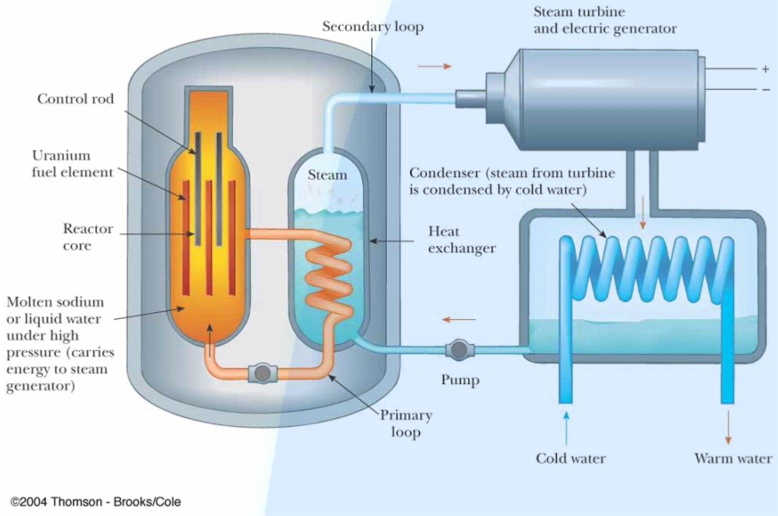
$$^{14}N + p \rightarrow {}^{15}O + \gamma$$

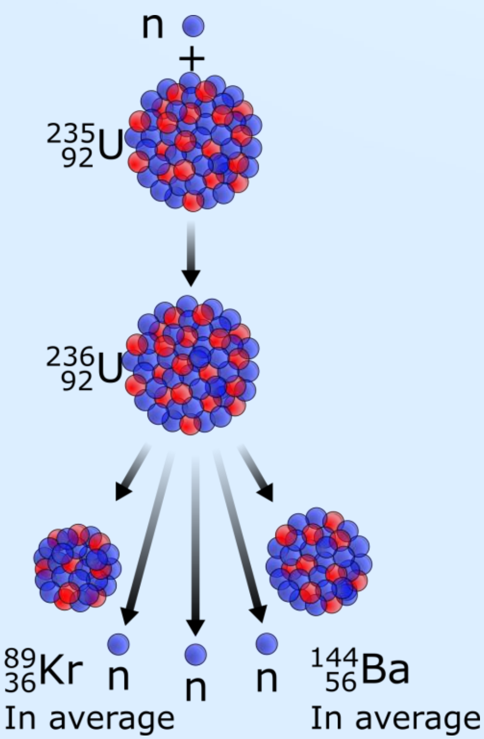
$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He$$

} CNO cycle

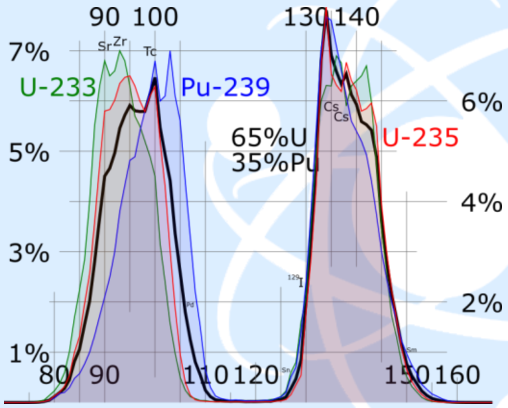


THIS SITUATION IS MUCH MORE CRITICAL AS $A \gg 1$



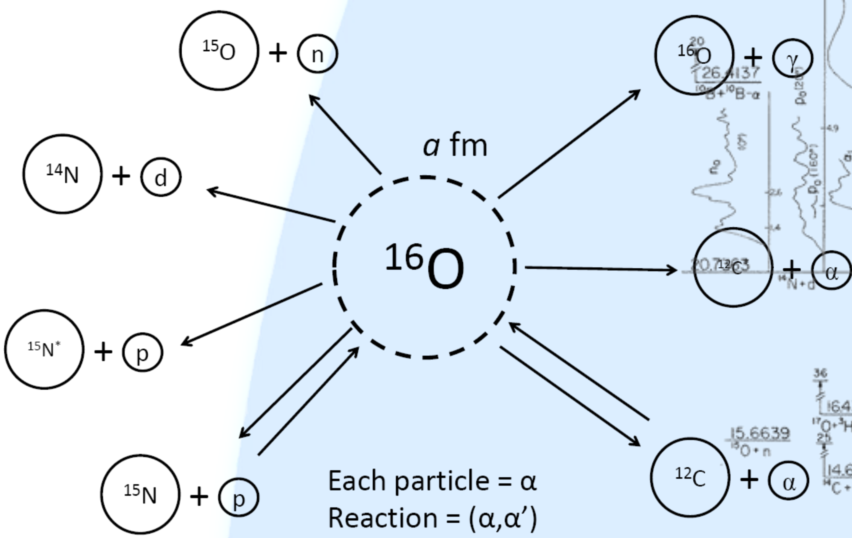


Neutron capture followed by fission

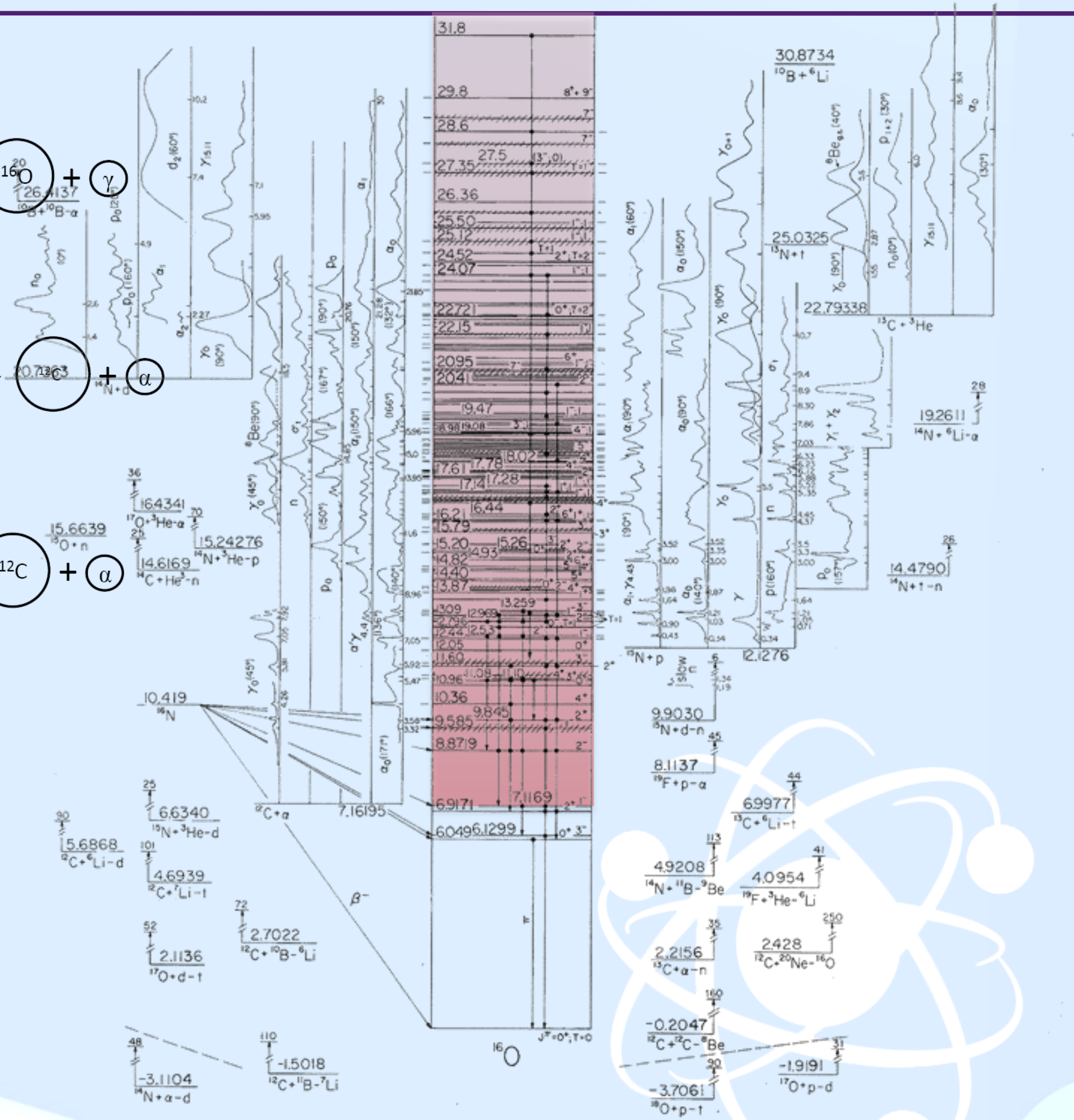


The understanding of reaction channels competition is crucial for instance to predict the production of long-lived radioactive isotopes.

« SIMPLE CASE »: REMINDER



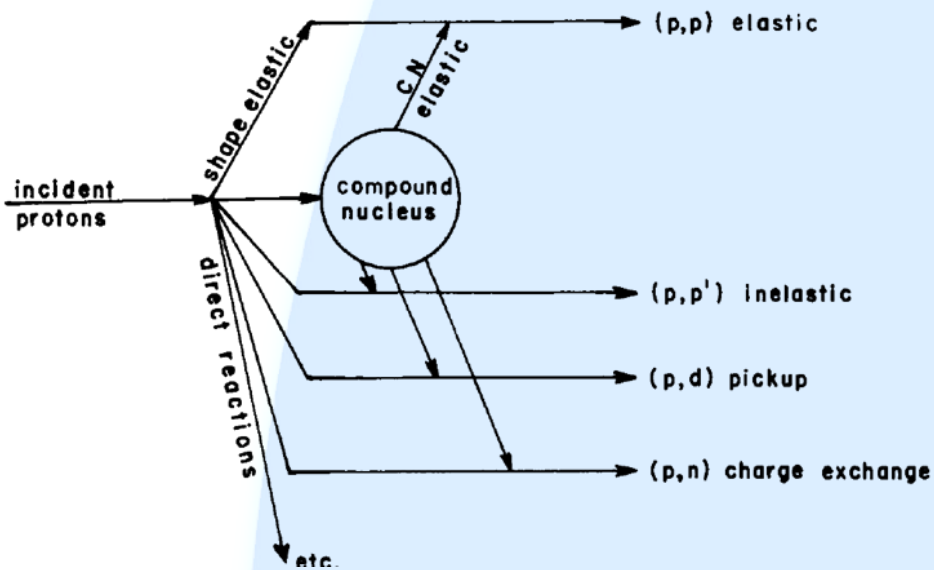
Each particle = α
Reaction = (α,α')



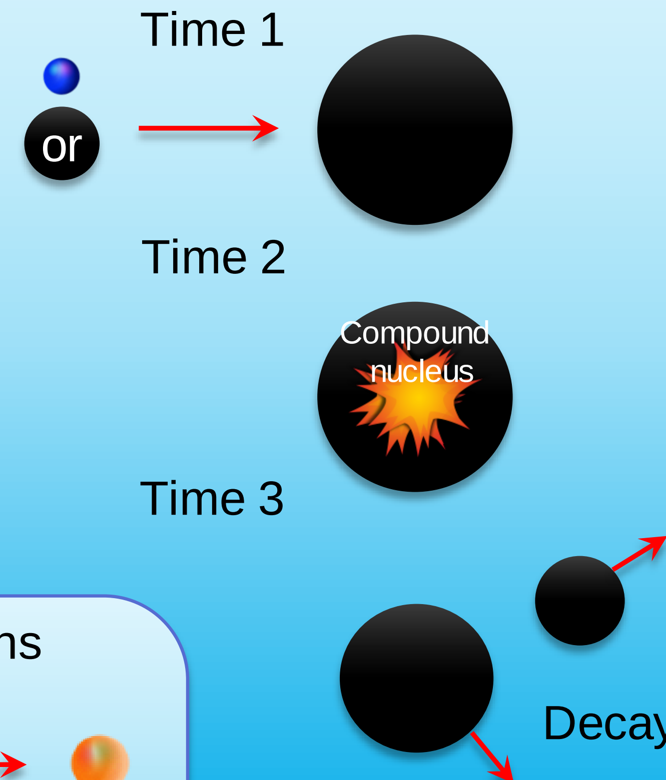
Open channel depends on the masses of the neighbouring nuclei and light fragments

NUCLEON-NUCLEUS COLLISIONS
THE MANY-FACETS OF A NUCLEON-INDUCED REACTION

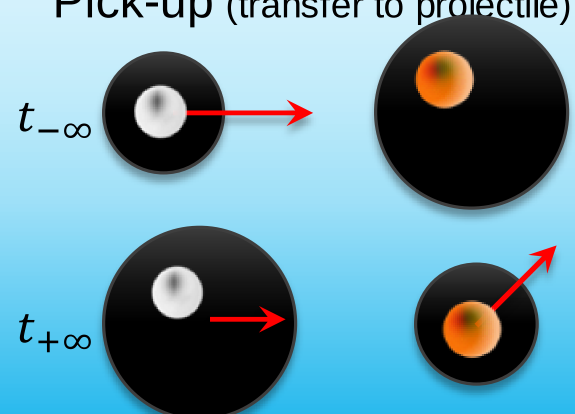
IJCLab Irène Joliot-Curie
Laboratoire de Physique des 2 Infinis



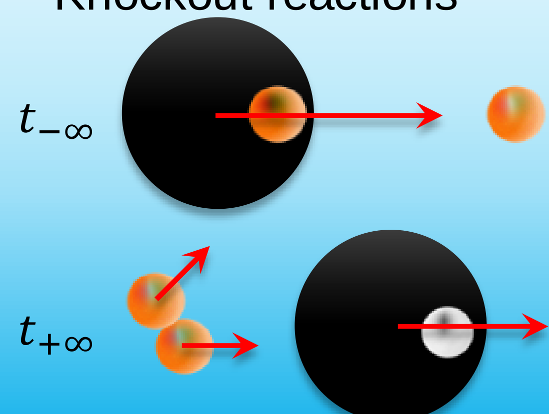
Compound nucleus formation



Pick-up (transfer to projectile)



Knockout reactions



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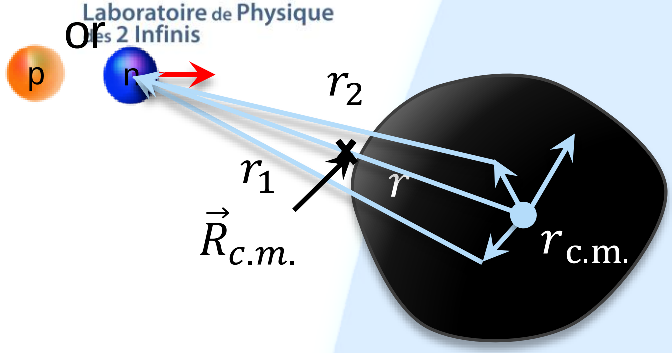
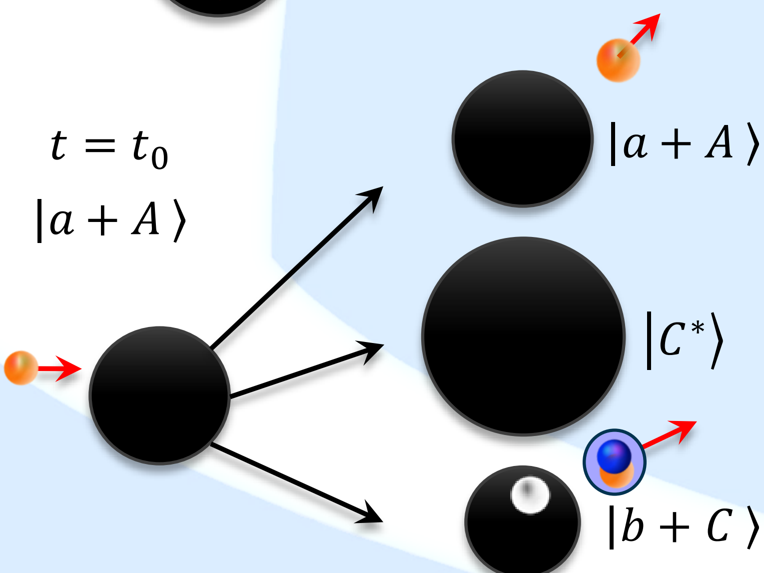
WHAT ARE THE NEW ASPECTS COMPARED TO THE BINARY ELASTIC CASE





1. The diffusion is made by an extended source, i.e. the target/projectile have an internal degrees of freedom [≠point like]
   - ➢ A general scheme should be developed
2. We need to describe the internal structure and excitations of the target/projectile
   - ➢ Exciting the target will induce an energy loss of projectile kinetic energy – inelastic scattering
3. The different channels compete and will interfere leading to modified cross-sections
   - ➢ We need to develop a general scheme able to describe the competition between several channels
   - ➢ Or effectively accounting for reduction to one channel contribution to σ

A few formal aspects

The goal 1) general scheme with approximation 2) effective description for an ad hoc modeling

FORMAL ASPECTS

The general form Schrödinger equation

with $E = \frac{\hbar^2 k^2}{2\mu}$

$$(1) \quad \longrightarrow \quad (\Delta + k^2)\varphi(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})\varphi(\vec{r})$$

We know the solution for $V = 0$

$$(\Delta + k^2)\varphi_0(\vec{r}) = 0 \text{ where } \varphi_0(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

We solve the differential equation with the green function method

$$(\Delta + k^2)\varphi_0(\vec{r}) = 0$$

The free particle green function is defined by

$$(\Delta + k^2)G_0(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}')$$

This equation has two solutions

$$G_0^{\pm}(\vec{r} - \vec{r}') = -\frac{e^{\pm ik|r - r'|}}{4\pi|r - r'|}$$

are called outgoing incoming Green functions

The general solution of (1) is

$$\varphi^{\pm}(\vec{r}) = \varphi_0(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_0^{\pm}(\vec{r} - \vec{r}')V(\vec{r}')\varphi^{\pm}(\vec{r}')$$

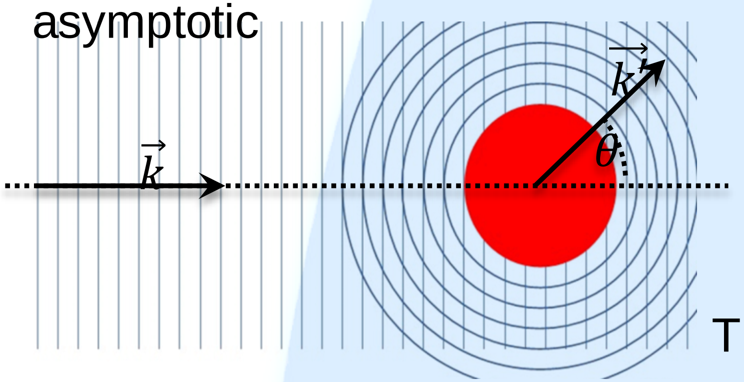
Called “Integral form of the diffusion equation”

We can also defined the full green function $G(\vec{r} - \vec{r}')$, solution of $\left(\Delta + k^2 - \frac{2\mu}{\hbar^2}V(\vec{r})\right)G = \delta(\vec{r} - \vec{r}')$ then we have $\varphi = [1 - GV]\varphi_0$ “matrix form”

FORMAL ASPECTS

$$\varphi^{\pm}(\vec{r}) = \varphi_0(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_0^{\pm}(\vec{r}-\vec{r}')V(\vec{r}')\varphi^{\pm}(\vec{r}')$$

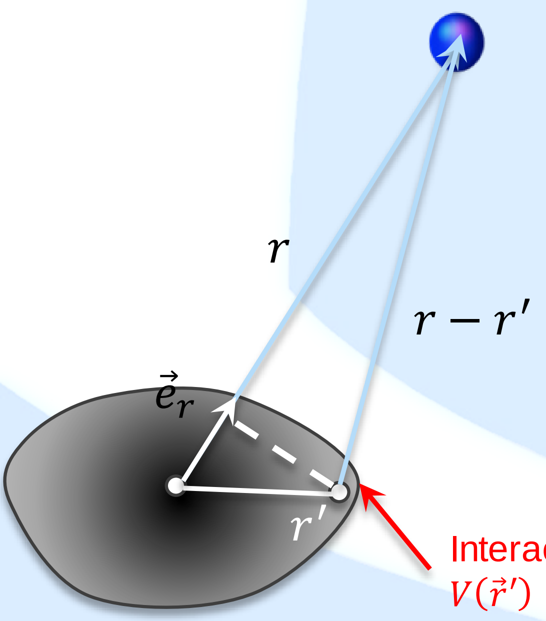
Recovering large distance asymptotic



For the diffusion scattering problem

$$\psi_{\boldsymbol{k}}^{\pm}(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} - \frac{2\mu}{\hbar^2}\int d^3r' \frac{e^{\pm ik|r-r'|}}{4\pi|r-r'|}V(\vec{r}')\psi_{\boldsymbol{k}}^{\pm}(\vec{r}')$$

This is the Lippmann-Schwinger equation



At large distance $(r \gg 1)$: $|\vec{r}-\vec{r}'| \cong r - \vec{e}_r\cdot\vec{r}' + \cdots$

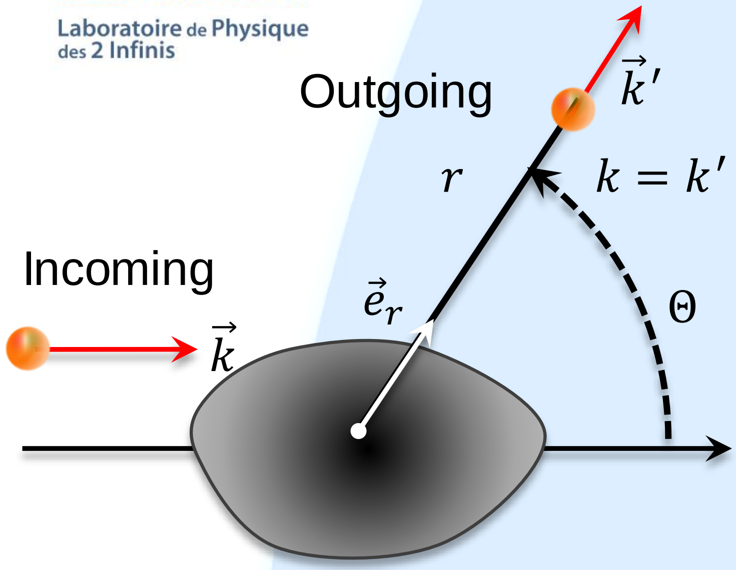
$$\frac{e^{\pm ik|r-r'|}}{|r-r'|} \cong \frac{e^{\pm ikr}}{r}e^{\mp i\vec{k}'\cdot\vec{r}'}$$ with $\vec{k}' = k\vec{e}_r$

$k = k'$

This can be cast into $A\left(e^{i\vec{k}.\vec{r}} + f(\Theta,\varphi)\frac{e^{ikr}}{r}\right)$ for $\psi_{\boldsymbol{k}}^{+}(\vec{r})$ if

$$f(\theta,\varphi) = -2\pi^2\frac{2\mu}{\hbar^2}\int d^3r' \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{(2\pi)^{3/2}}V(\vec{r}')\psi_{\boldsymbol{k}}^{+}(\vec{r}')$$

CROSS-SECTION AND T-MATRIX



From

$$f(\theta,\varphi) = -2\pi^2 \frac{2\mu}{\hbar^2} \int d^3r' \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{(2\pi)^{3/2}} V(\vec{r}')\psi_{\boldsymbol{k}}^{+}(\vec{r}')$$

We recognize

$$f(\theta,\varphi) = -2\pi^2 \frac{2\mu}{\hbar^2} \langle \varphi_{0,\boldsymbol{k}'} | V | \psi_{\boldsymbol{k}}^{+} \rangle$$

We deduce the differential cross-section as:

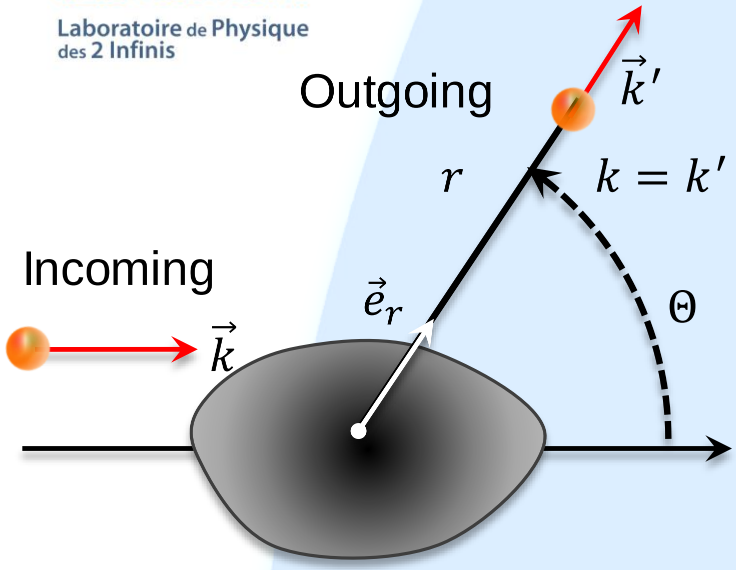
$$\frac{d\sigma(\theta,\varphi)}{d\Omega} = |f(\Theta,\varphi)|^2 = \left(2\pi^2 \frac{2\mu}{\hbar^2}\right)^2 \left|\langle \varphi_{0,\boldsymbol{k}'} | V | \psi_{\boldsymbol{k}}^{+} \rangle\right|^2 = \left(2\pi^2 \frac{2\mu}{\hbar^2}\right)^2 \left|T_{\boldsymbol{k}',\boldsymbol{k}}\right|^2$$

$T_{\boldsymbol{k}',\boldsymbol{k}}$ is the on-shell $[k = k']$ T-matrix element and relates to the S-matrix by

$$S_{\boldsymbol{k}',\boldsymbol{k}} = \delta(\vec{k} - \vec{k}') - 2\pi\delta(E_k - E_{k'}) T_{\boldsymbol{k}',\boldsymbol{k}}$$

From the definition of $\psi_{\boldsymbol{k}}^{\pm}(\vec{r})$, we notice that $S_{\boldsymbol{k}',\boldsymbol{k}} = \langle \psi_{\boldsymbol{k}'}^{-} | \psi_{\boldsymbol{k}}^{+} \rangle$

T-MATRIX, PHASE SHIFTS AND POTENTIAL



Starting from

$$S_{\boldsymbol{k}',\boldsymbol{k}} = \delta(\vec{k} - \vec{k}') - 2\pi\delta(E_k - E_{k'})T_{\boldsymbol{k}',\boldsymbol{k}}$$

We can perform a partial wave decomposition to obtain

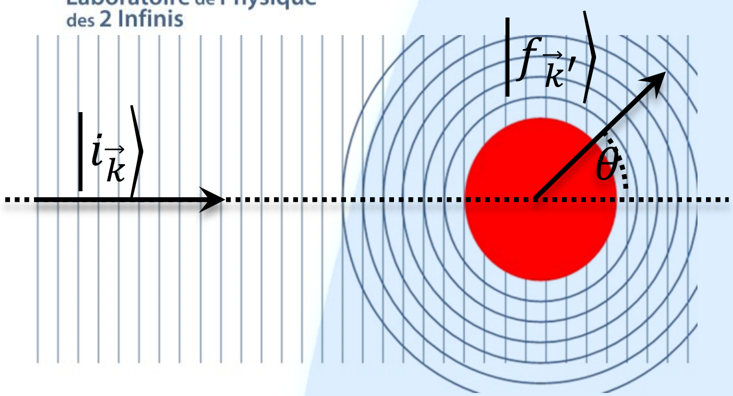
$$S_l = 1 - 2\pi i T_l(E)$$

So $T_l(E) = -{}^{1}/_{\pi}\, e^{i\delta_l(E)} \sin\delta_l(E)$. Similarly

$$T_l(E) = \frac{2\mu}{\pi\hbar^2}\int dr\, rJ_l(kr)\, V(r)u_l(r)$$

So we have

$$e^{i\delta_l(E)}\sin\delta_l(E) = -\frac{2\mu}{\hbar^2}\int dr\, rJ_l(kr)\, V(r)u_l(r)$$



We have solved

$$(\Delta + k^2)\varphi(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})\varphi(\vec{r})$$

And kept only the solution $\psi_{\boldsymbol{k}}^{+}$ corresponding to an incoming plane wave of momentum $\vec{k}$

At large distance, where reaction channels are defined, we find:

$$\psi_{\boldsymbol{k}}^{+}(\vec{r}) \overset{\infty}{\rightarrow} A\left(e^{i\vec{k}.\vec{r}} + f(\Theta,\varphi)\frac{e^{ikr}}{r}\right)$$

This can be cast into the form

$$|\psi_{\boldsymbol{k}}^{+}\rangle = | i_{\vec{k}}\rangle\langle i_{\vec{k}}|\psi_{\boldsymbol{k}}^{+}\rangle + \sum | f_{\vec{k}'}\rangle\langle f_{\vec{k}'}|\psi_{\boldsymbol{k}}^{+}\rangle$$

So that the probability to populate a given exit channel from the entrance channel is $\left|f_{i\rightarrow f}\right|^2 = \left|\langle f_{\vec{k}'}|\psi_{\boldsymbol{k}}^{+}\rangle\right|^2$, such that the differential cross section is

$$\frac{d\sigma_{i\rightarrow f}}{d\Omega} \propto \left|\langle f_{\vec{k}'}|\psi_{\boldsymbol{k}}^{+}\rangle\right|^2$$

Lippmann-Schwinger equation [in any of their form] are particularly useful:

$$\boxed{\varphi^{\pm}(\vec{r})} = \varphi_0(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_0^{\pm}(\vec{r}-\vec{r}')V(\vec{r}')\boxed{\varphi^{\pm}(\vec{r}')}$$

The equation is self consistent and can be used to write $\varphi^{\pm}$ as a series [perturbative expansion with V]

Illustration: perturbative expansion

1. At zeroth order in $V(r)$, the scattering wavefunction translates to unperturbed incident plane wave that is $\varphi^{\pm}(\vec{r}) = \varphi_0(\vec{r})$
2. At first order in V, we find

$$\varphi_{(1)}(\vec{r}) = \varphi_0(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_0^{\pm}(\vec{r}-\vec{r}')V(\vec{r}')\varphi_{(0)}(\vec{r}')$$

3. And then at second order

$$\varphi_{(2)}(\vec{r}) = \varphi_0(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_0^{\pm}(\vec{r}-\vec{r}')V(\vec{r}')\varphi_{(1)}(\vec{r}') \quad \cdots$$

REPRESENTATION OF THE BORN SERIES

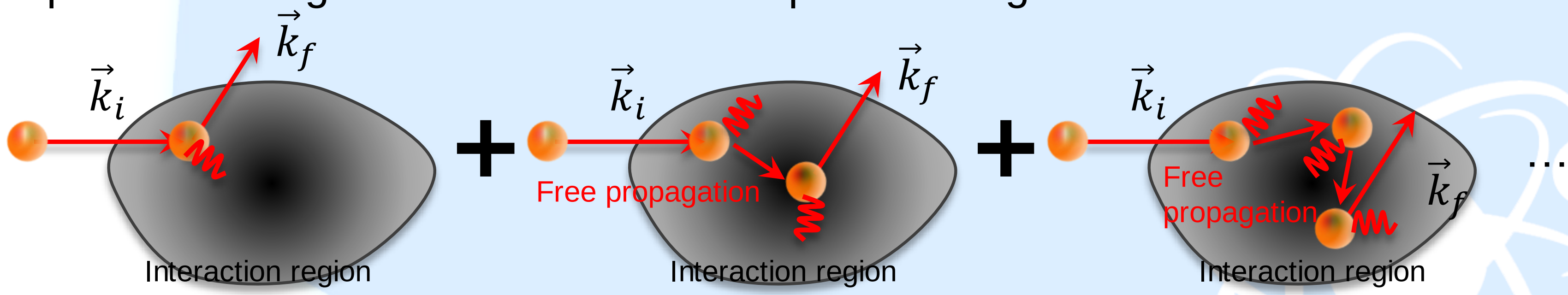
Formally the series writes

$$|\varphi_{\boldsymbol{k}}^{+}\rangle = |\varphi_{0,\boldsymbol{k}}\rangle + \frac{2\mu}{\hbar^2} G_0 V|\varphi_{0,\boldsymbol{k}}\rangle + \left(\frac{2\mu}{\hbar^2}\right)^2 G_0 V G_0 V|\varphi_{0,\boldsymbol{k}}\rangle + \cdots = \sum \left(\frac{2\mu}{\hbar^2} G_0 V\right)^n |\varphi_{0,\boldsymbol{k}}\rangle$$

Writing the scattering amplitude expressed as a Born series expansion we have

$$f(\Theta,\varphi) = -2\pi^2 \left\langle \varphi_{0,\boldsymbol{k}'} \middle| V \sum \left(\frac{2\mu}{\hbar^2} G_0 V\right)^n \middle| \varphi_{0,\boldsymbol{k}} \right\rangle$$

We can understand that the unperturbed plane wave undergoes a sequences of multiples scattering events from inside the potential region:



But the series may not converged until all terms are including if the potential is strong enough

THE BORN APPROXIMATION (I.E. FIRST ORDER)

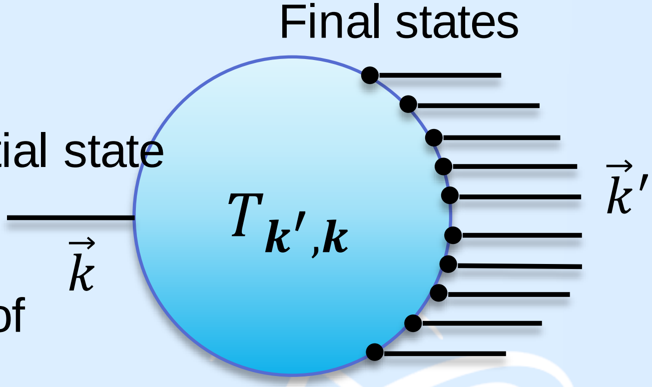
The leading term of the Born series is

$$f(\Theta,\varphi) = -2\pi^2 \left\langle \varphi_{0,\boldsymbol{k}'} \middle| \frac{2\mu}{\hbar^2} V \middle| \varphi_{0,\boldsymbol{k}} \right\rangle$$

Which gives

Unperturbed w.f.

$$\frac{d\sigma(\theta,\varphi)}{d\Omega} = |f(\Theta,\varphi)|^2 \propto \langle \varphi_{0,\boldsymbol{k}'} | V | \varphi_{0,\boldsymbol{k}} \rangle$$



At first order, the fermi golden rule is equivalent to the born approximation

$$\Gamma_{k\to k'} = \sum_{k'\in d\Omega} \frac{2\pi}{\hbar} |\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle|^2 \delta(E_k - E_{k'})$$

$$= \frac{2\pi}{\hbar} |\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle|^2 g(E_k)$$

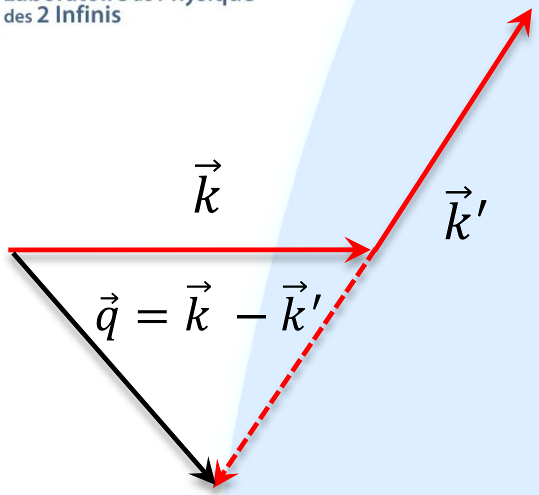
Density of state

Similarly, we can get the phase shift of the Born approximation

$$e^{i\delta_l(E)} \sin\delta_l(E) = -\frac{2\mu}{\hbar^2} \int dr\, r^2 J_l(kr)^2 V(r)$$

In particular it tells us that $\mathrm{sign}(\mathrm{V}) = \mathrm{sign}(\delta)$

EXPLICIT FORM OF THE CROSS SECTION AT THE BORN APPROXIMATION



$\vec{q}$ is the momentum transfer to the target by the projectile

Starting from

$$f(\Theta,\varphi) = -2\pi^2 \left\langle \varphi_{0,\boldsymbol{k}'} \middle| \frac{2\mu}{\hbar^2} V \middle| \varphi_{0,\boldsymbol{k}} \right\rangle$$

We immediately obtain

$$f_{\text{Born}}(\Theta,\varphi) = -2\pi^2 \frac{2\mu}{\hbar^2} \int d^3r' \frac{e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}}{(2\pi)^3} V(\vec{r}')$$

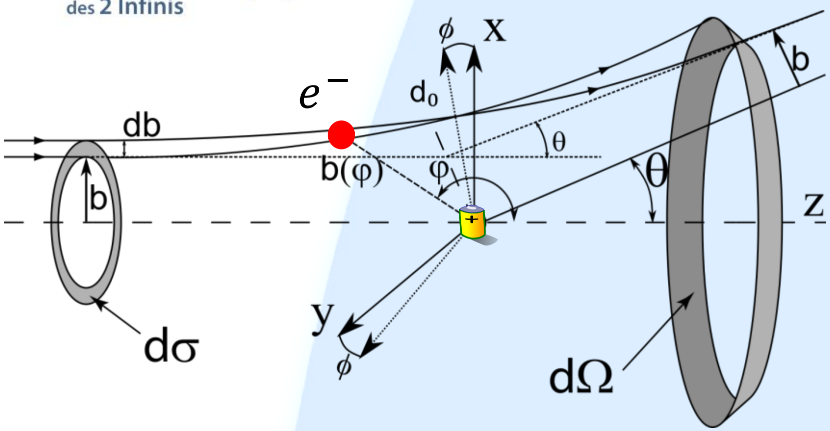
Which is nothing but the 3D Fourier transform of the potential

If the potential is spherical symmetric

$$f_{\text{Born}}(\Theta,\varphi) = -\frac{2\mu}{\hbar^2} \int r'^2 dr' \frac{\sin(qr')}{qr'} V(r')$$

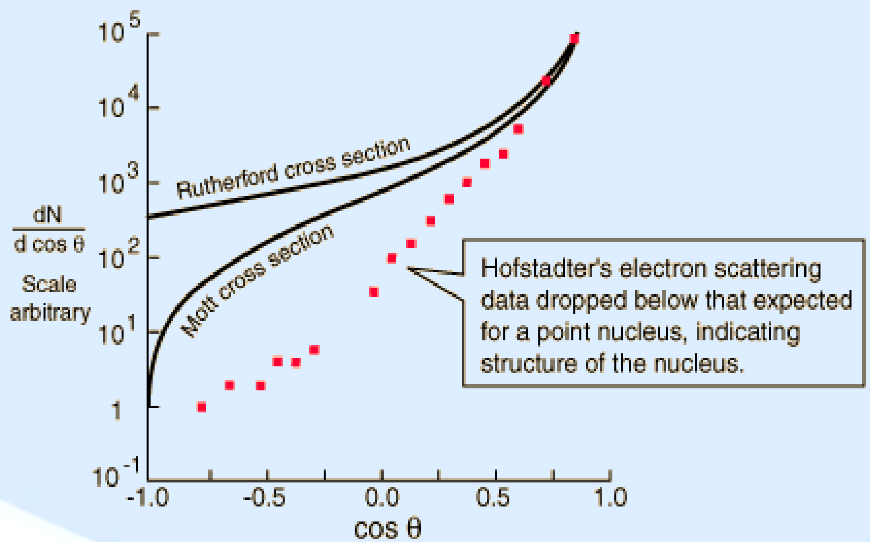
With $q^2 = k^2 + k'^2 - 2kk' \cos\theta$

BORN APPROXIMATION: APPLICATION TO THE ELECTRON SCATTERING CASE



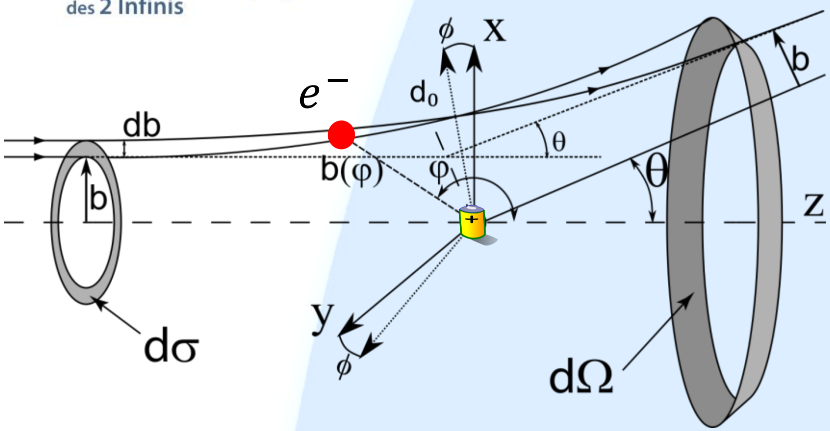
Classical approximation to the scattering with relativistic correction

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \simeq \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}} \cos^2\frac{\Theta}{2}$$



Hofstadter, R., *et al.*, Phys. Rev. 92, 978 (1953).

BORN APPROXIMATION: APPLICATION TO THE ELECTRON SCATTERING CASE



Quantum Scattering by a point like particle with a Yukawa or Coulomb potential

$$f_{\text{Born}}(\Theta,\varphi) = -\frac{2\mu}{\hbar^2}\int r'^2 dr' \frac{\sin(qr')}{qr'} V(r')$$

If we assume that $V(r) = -V_0\, {e^{-\alpha r}}/{r}$ then

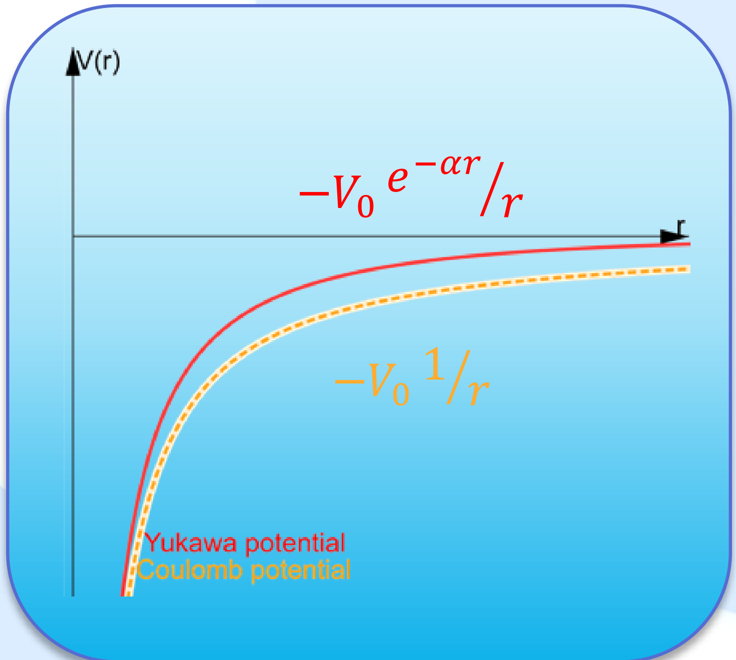
$$f_{\text{Born}}(\Theta,\varphi) = \frac{2\mu}{\hbar^2} V_0 \frac{1}{\alpha^2 + q^2}$$

For the Coulomb case we take the limit $\alpha \longrightarrow 0$, and in the elastic case $k = k'$ thus

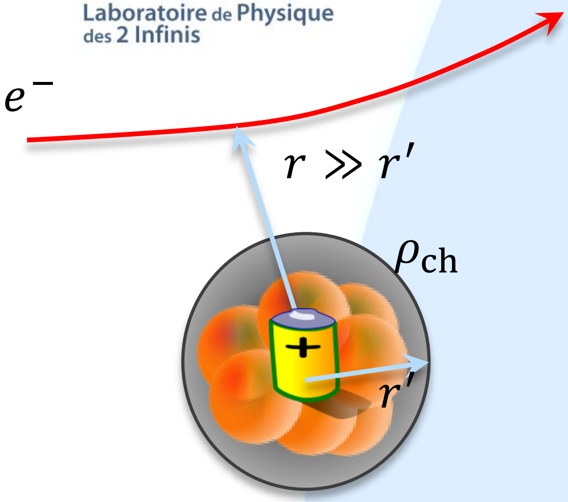
$$q^2 = 2k^2(1-\cos\theta) = 4k^2 \sin^2 {\theta}/{2}$$

We recover the classical formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{8\pi\varepsilon_0 m v_0^2}\right)^2 \csc^4\left(\frac{\theta}{2}\right)$$



BORN APPROXIMATION: APPLICATION TO THE ELECTRON SCATTERING CASE



At $r \gg r'$, the potential felt by e^- is given by the convolution product between the charged density of the nucleus (proton density) and the coulomb potential

$$V(r) = Z_1 e^2 \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}' - \vec{r}\,|}$$

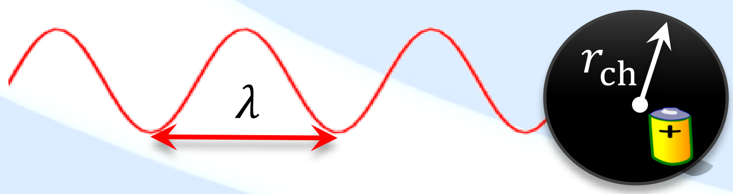
Note $\int d^3r' \rho(\vec{r}') = Z$

Since the Born scattering amplitude is a Fourier transform of the potential the cross section is a product of the charged density FT and the Coulomb potential FT (i.e. Rutherford) that is

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} [F(q)]^2$$

$F(q)$ is called the form factor

At low momentum transfer, $F(q) \sim 1 - \frac{1}{6} q^2 \langle r_{\text{ch}}^2 \rangle + \cdots$

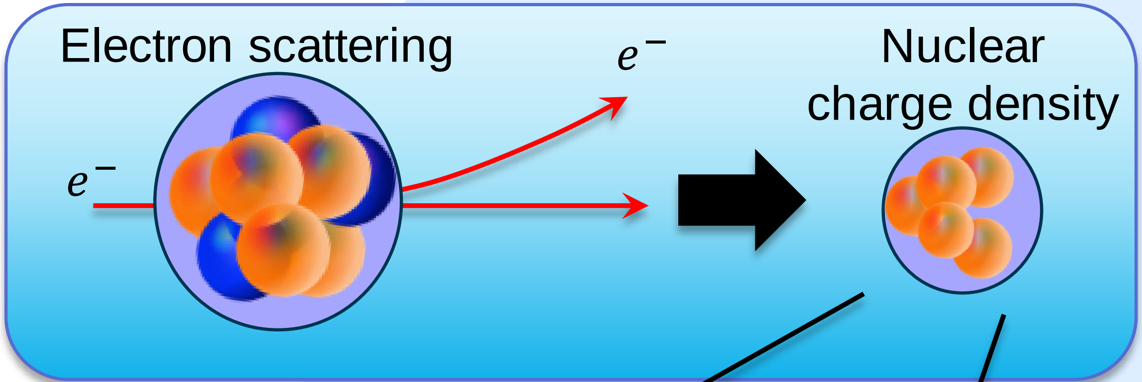


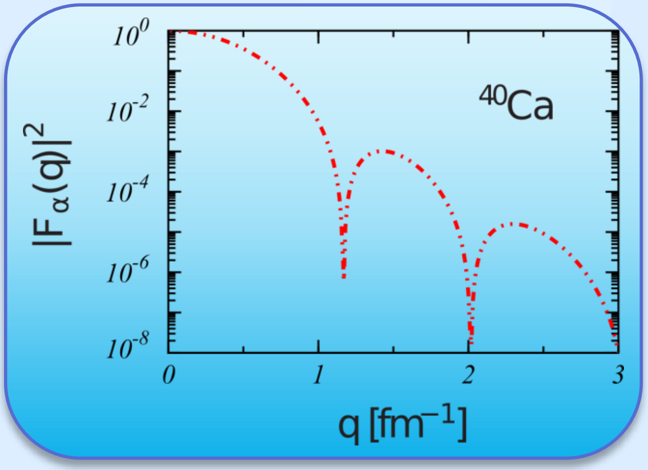
Note that

$$\lambda_{\text{Broglie}}^{e^-} \cong \frac{5 \cdot 10^3}{\sqrt{E}}\ \text{fm}$$

$$\lambda_{\text{Broglie}}^{N} \cong \frac{4{,}54}{\sqrt{E}}\ \text{fm}$$

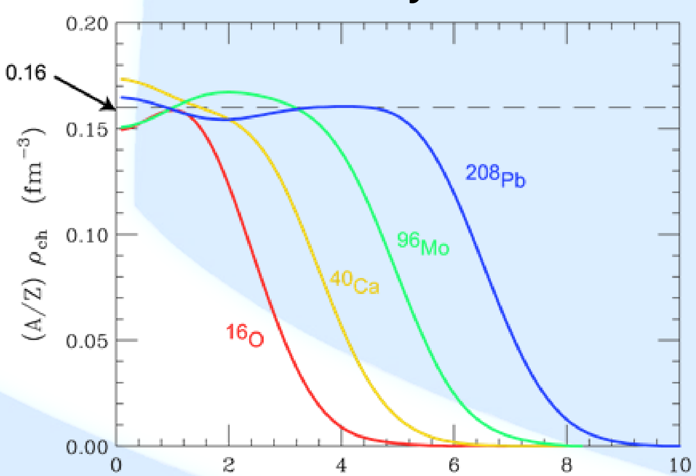
ELECTRON SCATTERING, INCOMPRESSIBILITY AND SATURATION





G. Co' *et al*, JCAP11(2012)010

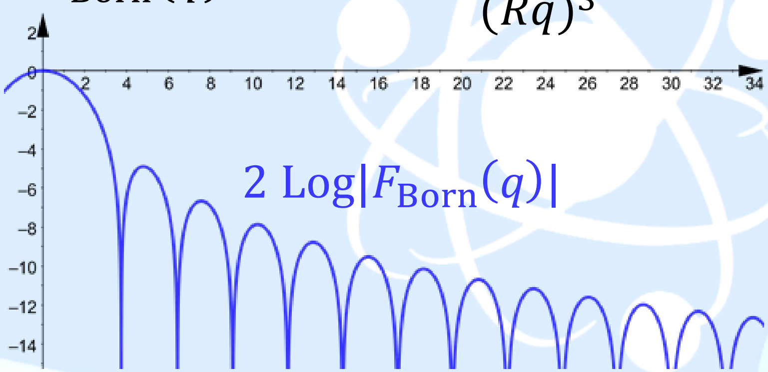
Density in the nucleus



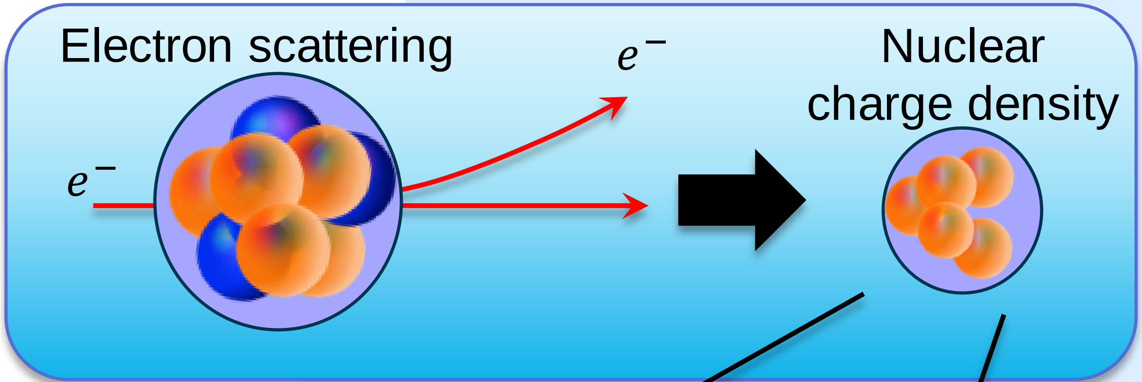
Large transferred momentum q provides shape of the central density distribution.

For uniform density

$$F_{\mathrm{Born}}(q) = 3\,\frac{\sin(Rq) - Rq\cos(Rq)}{(Rq)^3}$$

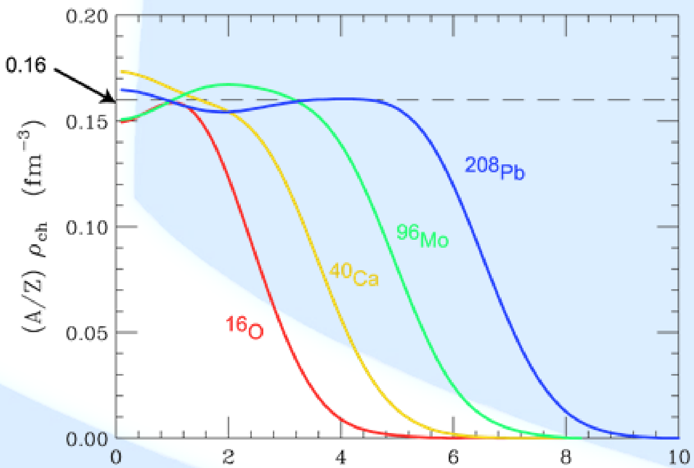


ELECTRON SCATTERING, INCOMPRESSIBILITY AND SATURATION

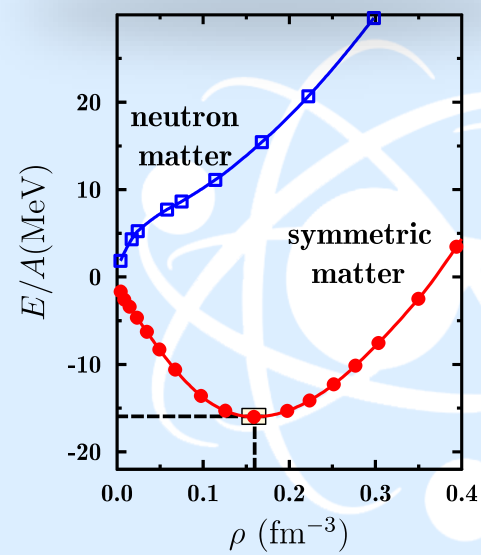


Nuclear behaves “like” incompressible Fermi systems with density

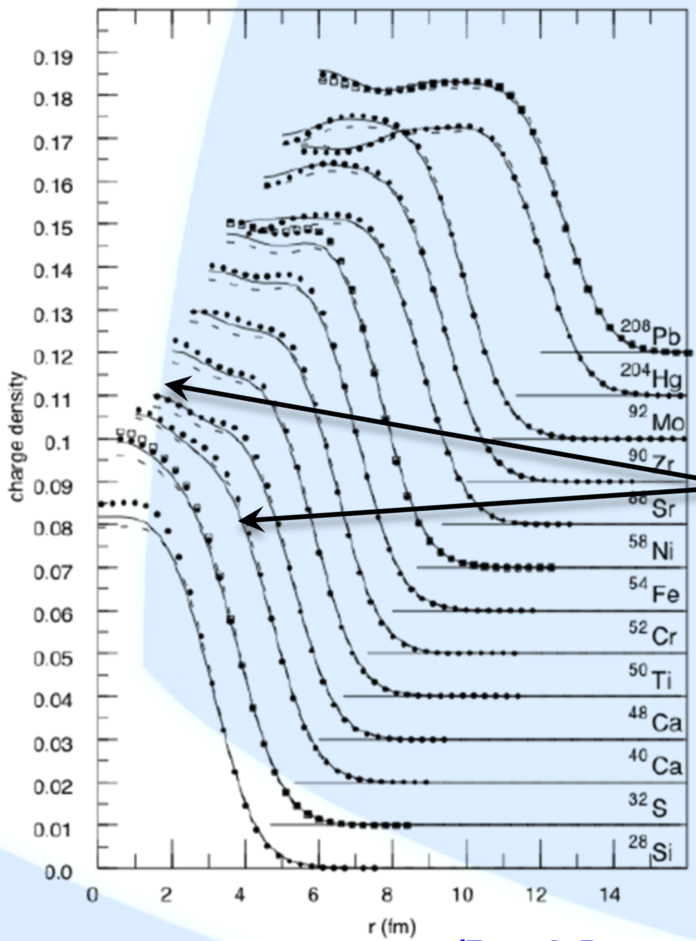
Density in the nucleus



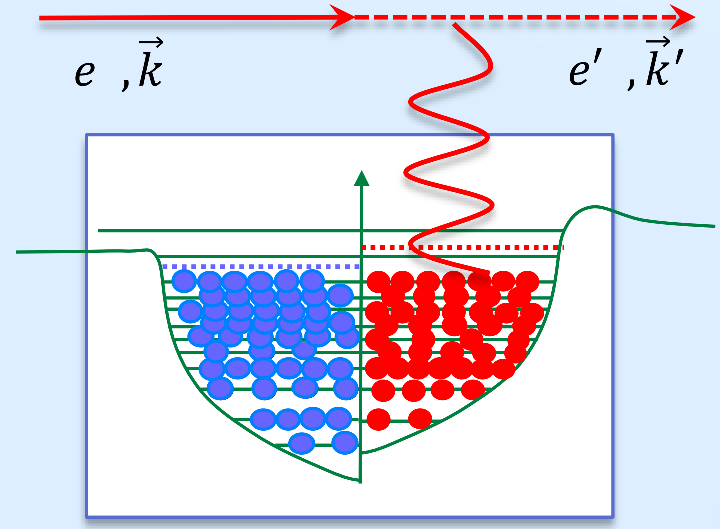
Infinite nuclear matter



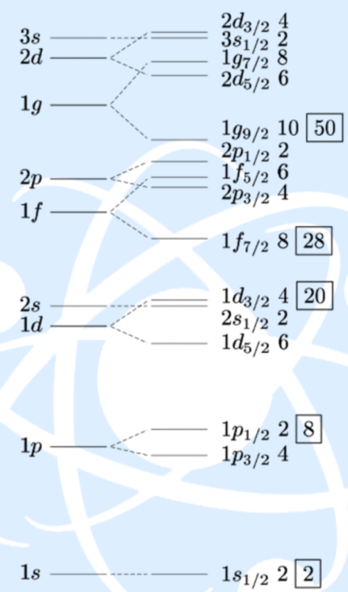
Systematic of nuclear charge density



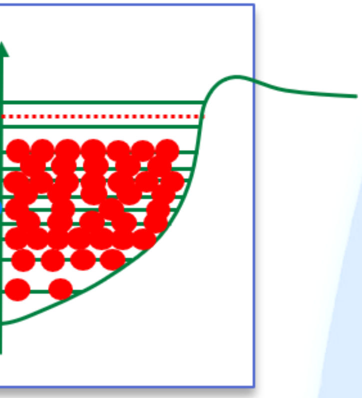
(From A. Brown website)



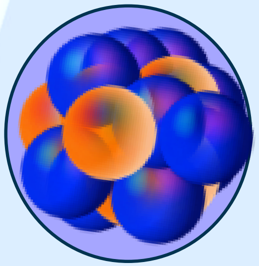
Oscillations probes shell effects and independent particle picture of the nucleus



DEPARTURE OF THE INDEPENDENT PARTICLE PICTURE



Density of ^{206}Pb



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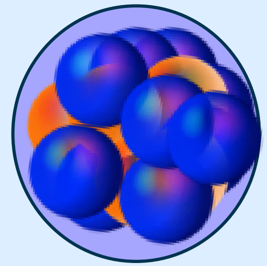
Density of ^{205}Tl



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“Wave-function” of the last proton

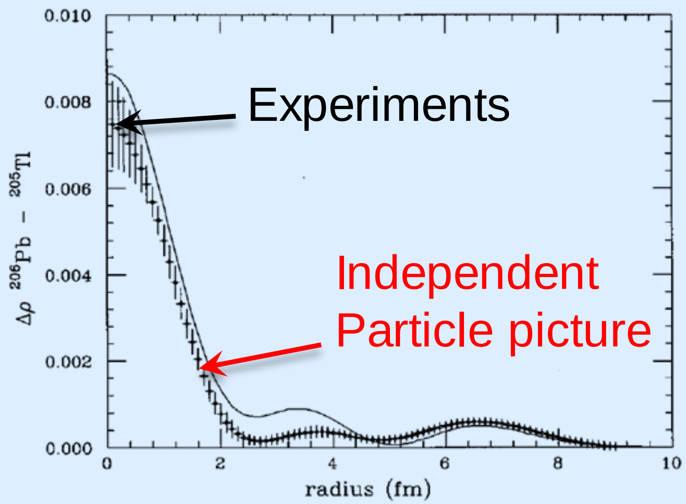


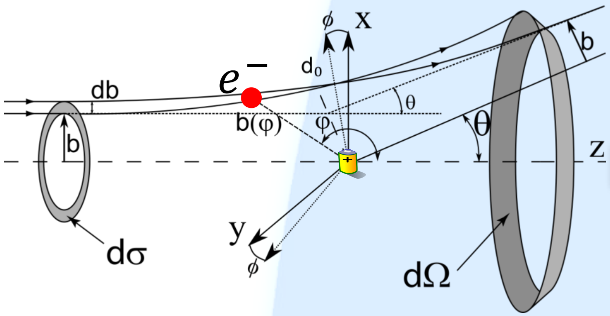
FIG. 3. Density difference between ^{206}Pb and ^{205}Tl. The experimental result of Cavendon *et al.* (1982) is given by the error bars; the prediction obtained using Hartree-Fock orbitals with adjusted occupation numbers is given by the curve. The systematic shift of 0.0008 fm^{-3} at $r \leqslant 4$ fm is due to deficiencies of the calculation in predicting the core polarization effect.

Departure from the Independent picture is observed: correlations are also important (CQFD)

From Pandharipande *et al*, Rev. Mod. Phys. 69, 981

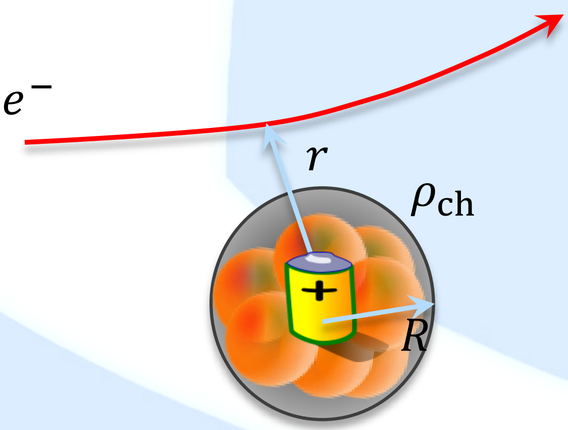
SOME CONCLUSIONS ON COULOMB INTERACTION WITH NUCLEI

From the simplest version Coulomb scattering, we have considered a series of reaction models of increasing complexity



Rutherford valid until matter wavelength probe nuclear effects

1. Nuclei are extended systems
2. Interact at short range with strong force

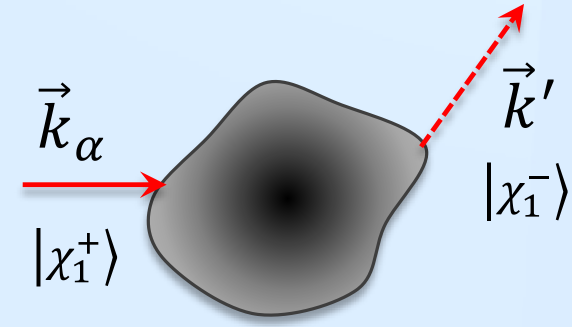


- Finite size extension
- Quantum corrections

1. Nuclear size
2. Densities
3. Correlations

A VARIANT OF THE BORN APPROXIMATION: THE DISTORTED WAVE-BORN APPROXIMATION (DWBA)

In the standard Born Approximation



$$\chi_k^+(\vec{r}) = \frac{e^{i\vec{k}_\alpha\cdot\vec{r}}}{(2\pi)^{3/2}} + \frac{2\mu}{\hbar^2}\int d^3r' \frac{e^{ik_\beta|r-r'|}}{4\pi|r-r'|} V(\vec{r}')\chi_k^+(\vec{r}')$$

Systematic constructive treatment

$$f = -\frac{2\mu}{4\pi\hbar^2}\langle \boldsymbol{k}'|V + VG_0V + \cdots|\chi_k^+\rangle$$

$$f_{\mathrm{Born}} = -\frac{2\mu}{4\pi\hbar^2}\langle \boldsymbol{k}'|V|\mathbf{k}\rangle$$

In some cases, the free-wave approximation is rather poor starting point.

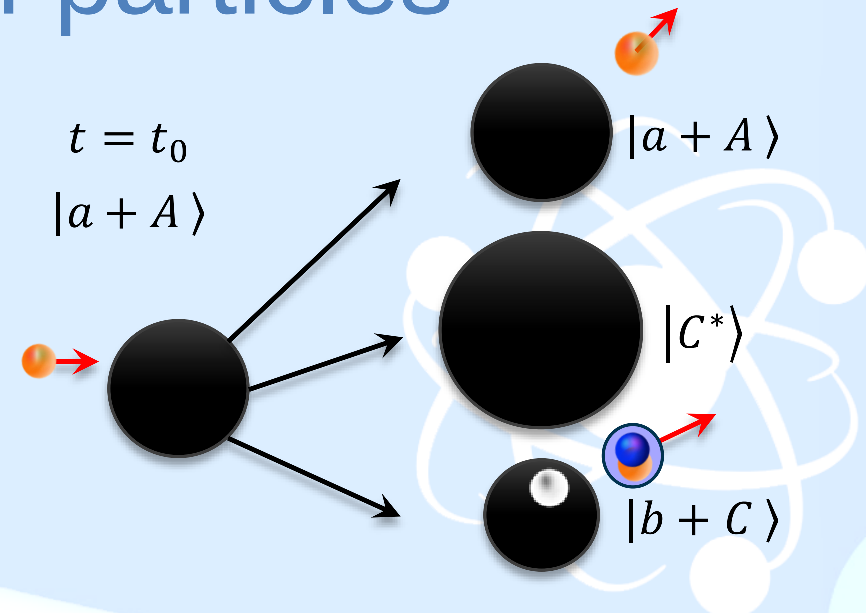
Suppose $V = V_{\mathrm{MF}} + V_{\mathrm{res}}$ and the solutions of $(\nabla^2 + k^2 - V_{\mathrm{MF}})\chi_1(\boldsymbol{k},\boldsymbol{r}) = 0$ are known/computable

One can show $f = f_1 - \frac{2\mu}{4\pi\hbar^2}\int d^3r'\chi_1^-(\boldsymbol{k},\vec{r})V_{\mathrm{res}}(\vec{r}')\chi_k^+(\vec{r}')$

The DWBA approximation consists in:

$\chi_k^+ \longrightarrow \chi_1^+(\boldsymbol{k},\boldsymbol{r})$ then $f = f_1 - \frac{2\mu}{4\pi\hbar^2}\langle\chi_1^-|V|\chi_1^+\rangle$

Competition between different channels and internal structure of particles



GENERALISATION TO MULTICHANNEL DECOMPOSITION

$$\psi_k^+(\vec{r}) \longrightarrow \left(e^{i\vec{k}.\vec{r}} + f(\Theta,\varphi)\frac{e^{ikr}}{r} \right)$$

$$f(\Theta,\varphi) = \sum_{\beta} f_\beta(\Theta,\varphi)$$

All energetically allowed opened channels β

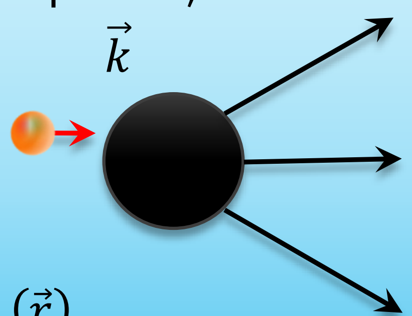
Note that closed channels also contribute but to reaction observable

$t = t_0$

Incoming wave

$|a + A\rangle$

$\vec{k}$



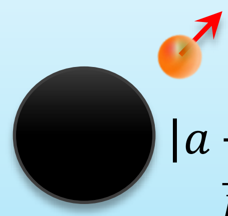
$\Psi_a \Psi_A \chi_k^i(\vec{r})$

$\chi_k^i(\vec{r}) \propto e^{i\vec{k}.\vec{r}}$

$H_a\Psi_a = E_0\Psi_a$
$H_A\Psi_A = E_0\Psi_A$

Ψ_a and Ψ_A quantum numbers defined conservation of total relative angular momentum

$t \gg t_0$



Elastic channel, always opened

$|a + A\rangle \; \Psi_a\Psi_A\chi_{k_\beta}^f(\vec{r})$

$\vec{k}_\beta$

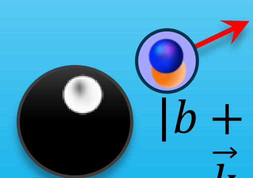
$H_a\Psi_a = E_0\Psi_a$
$H_A\Psi_A = E_0\Psi_A$

$|C^*\rangle \quad \Psi_{C^*}$

$\vec{k}_\beta$

$H_C\Psi_{C^*} = E_n\Psi_{C^*}$

channel Q-value



$\Psi_b\Psi_C\chi_{k_\beta}^f(\vec{r})$

$|b + C\rangle$

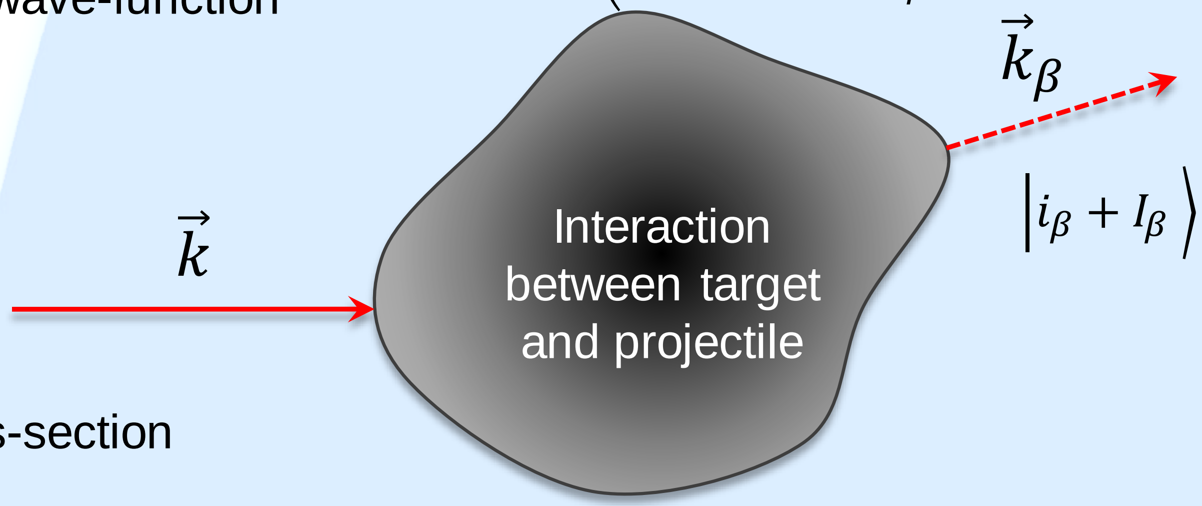
$H_b\Psi_b = E_0\Psi_b$
$H_C\Psi_C = E_0\Psi_C$

$\vec{k}_\beta$

SCATTERING PROBLEM OF COMPOSITE SYSTEMS

Scattering wave-function

$$\psi_k^+(\vec{r}) \longrightarrow \left(e^{i\vec{k}.\vec{r}}\Psi_a\Psi_A + \sum_\beta f_\beta(\Theta,\varphi)\frac{e^{ik_\beta r}}{r}\Psi_{i_\beta}\Psi_{I_\beta} \right)$$



Partial cross-section

$$\frac{d\sigma_\beta}{d\Omega} = \frac{\vec{j}_f \cdot d\vec{S}/_{r^2}}{\vec{j}_i \cdot \hat{k}}$$

Since $\vec{j} = \rho\vec{v}$ with $\vec{v}$ the wave vector, we have that

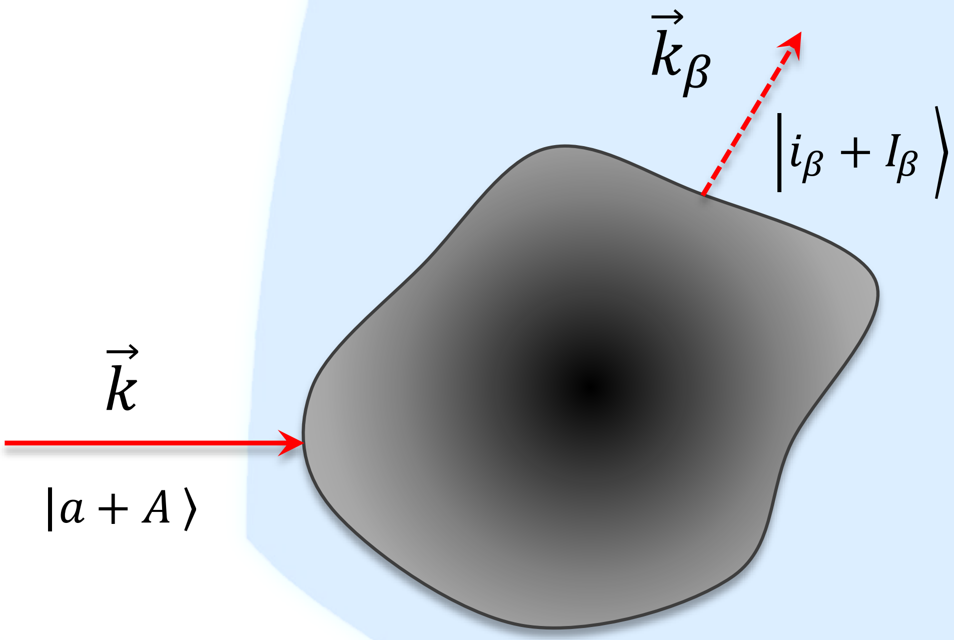
$$\frac{d\sigma_\beta}{d\Omega} = \frac{v_\beta}{v}\left|f_\beta(\Theta,\varphi)\right|^2$$

Elastic scattering $\frac{v_\beta}{v} = 1$

Inelastic scattering $\frac{v_\beta}{v} \neq 1$

SCATTERING PROBLEM OF COMPOSITE SYSTEMS

$$\frac{d\sigma_\beta}{d\Omega} = \frac{v_\beta}{v}\left|\breve{f}_\beta(\Theta,\varphi)\right|^2$$



$|a + A\rangle$

Elastic channel with $\frac{v_\beta}{v} = 1$, always opens

$\left|a^* + A^*\right\rangle$

Inelastic scattering $\frac{v_\beta}{v} \neq 1$, Energetically opens if $E_{\text{c.m.}}$ is greater than the reaction threshold

$|b + C\rangle$

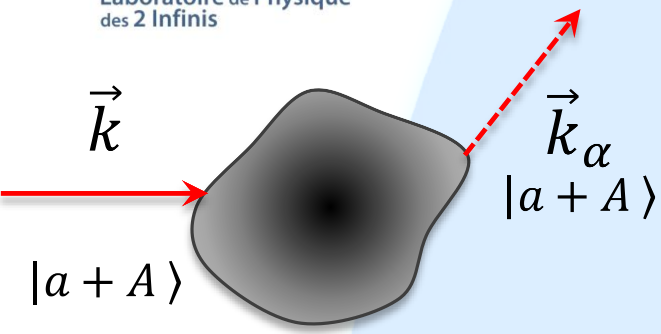
$|b + d + C\rangle$

All other reaction channels energetically allowed ($\frac{v_\beta}{v} \neq 1$)

Non-elastic channels

Channels will all interfere…

INFLUENCE OF THE NON-ELASTIC CHANNELS ON CROSS-SECTION



With only elastic channel

$$\psi_k^+(\vec{r}) \longrightarrow \left(e^{i\vec{k}.\vec{r}} + f(\Theta, \varphi) \frac{e^{ik_\alpha r}}{r} \right)$$

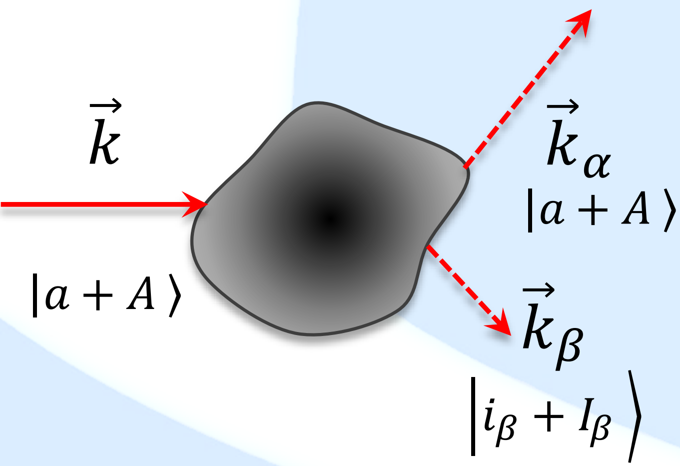
$$u_{\alpha,l}(r > \mathrm{R}) = A_{\alpha,l}\rho\left(H_l^-(\rho) - S_{\alpha,l}H_l^+(\rho)\right)$$

The conservation of the momentum leads to $k = k_\alpha$, the conservation of the flux [which implies the unitarity of the S-matrix i.e. $S_{\alpha,l}S_{\alpha,l}{}^* = 1$] means $S_{\alpha,l} = e^{2i\delta_l}, \delta \in \mathbb{R}$

Adding non-elastic channels

$$u_{\alpha,l}(r > \mathrm{R}) = A_{\alpha,l}\rho\left(H_l^-(\rho) - S_{\alpha,l}H_l^+(\rho)\right)$$

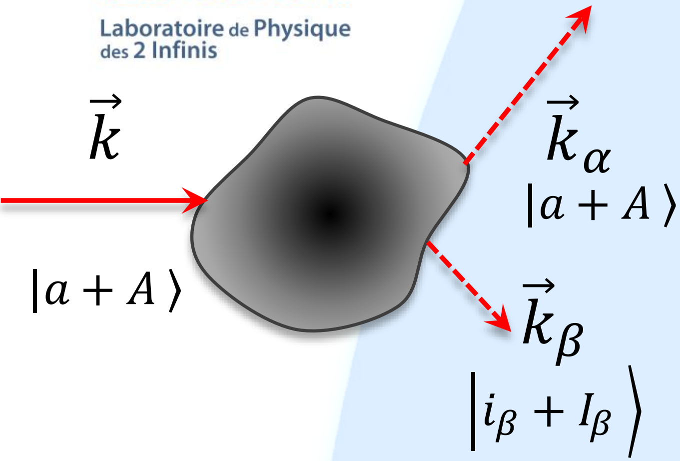
$$u_{\beta,l}(r > \mathrm{R}) = -A_{\beta,l}\rho S_{\beta,l}H_l^+(\rho)$$



Where $S_{\beta,l} = \sqrt{v/v_\beta}\, \tilde{S}_{\beta,l}$ and $\tilde{S}_{\beta,l} = e^{2i\delta_l}, \delta \in \mathbb{C}$. Total energy is conserved but $k_\beta \neq k$ due to energy consumed by the Q value. The flux is distributed among channels:

$$\left|S_{\alpha,l}(E)\right|^2 + \sum \left|S_{\beta,l}(E)\right|^2 = 1$$

ELASTIC, REACTION AND TOTAL CROSS SECTION



$$\psi_k^+(\vec{r}) \longrightarrow \left(e^{i\vec{k}.\vec{r}} + \sum \tilde{f}_\beta(\Theta,\varphi)\,\frac{e^{ik_\beta r}}{r} \right)$$

Elastic channel:

$$\sigma_{\mathrm{el}} = \frac{\pi}{k^2} \sum (2l+1) \left| 1 - \tilde{S}_{\alpha,l} \right|^2$$

Inelastic channels:

$$\sigma_{\mathrm{in}} = \frac{\pi}{k^2} \sum (2l+1) \left| \tilde{S}_{\beta,l} \right|^2$$

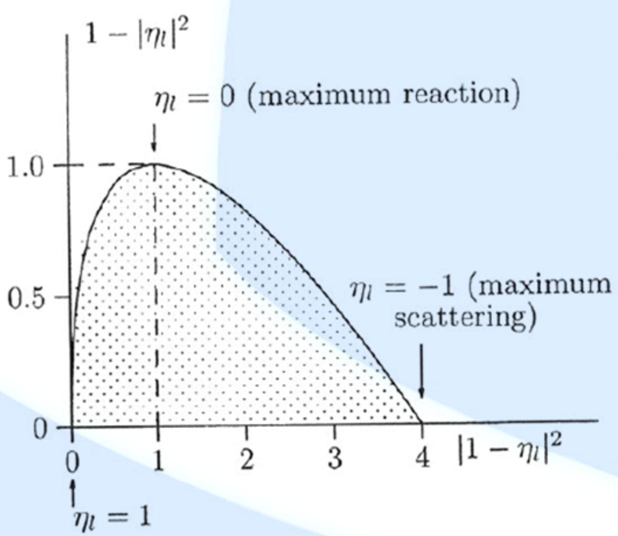
Sum of all inelastic channels (absorption cross-sec.):

$$\sigma_{\mathrm{abs}} = \frac{\pi}{k^2} \sum (2l+1) \left(1 - \left| \tilde{S}_{\alpha,l} \right|^2 \right)$$

from $\left| S_{\alpha,l} \right|^2 + \sum \left| S_{\beta,l} \right|^2 = 1$, and total cross-section

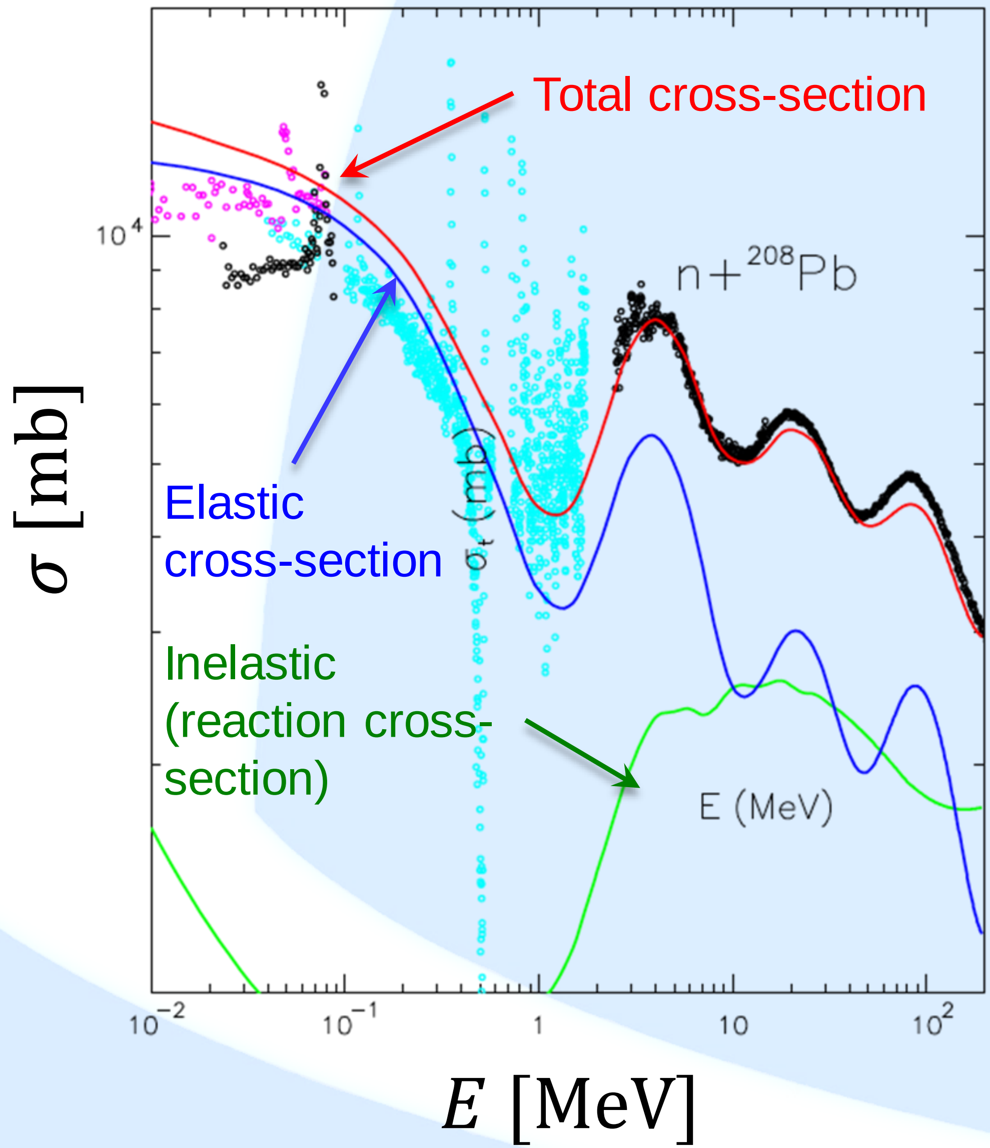
$$\sigma_{\mathrm{tot}} = \sigma_{\mathrm{el}} + \sigma_{\mathrm{abs}}$$

$$= \frac{2\pi}{k^2} \sum (2l+1) \left(1 - \mathrm{Re}(\tilde{S}_{\alpha,l}) \right)$$

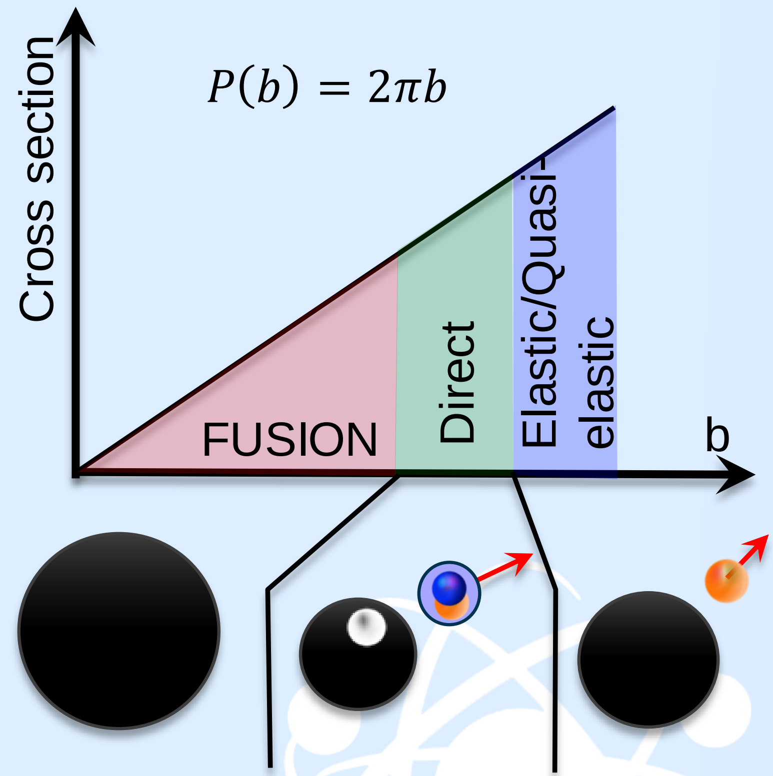


From Bertulani, Introduction to nuclear physics

ILLUSTRATION WITH THE NUCLEON-NUCLEUS CASE
ELASTIC, REACTION AND TOTAL CROSS SECTION

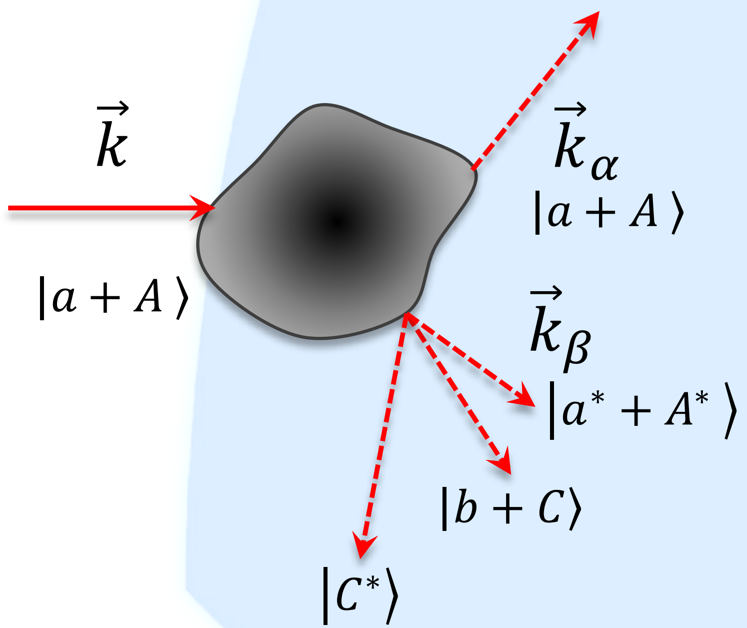


Why the elastic scattering dominates?



The more peripheral is the collision the highest is the associated cross-section

TOWARDS A SIMPLIFIED DESCRIPTION OF THE NUCLEON-NUCLEUS PROBLEM: OPTICAL POTENTIAL

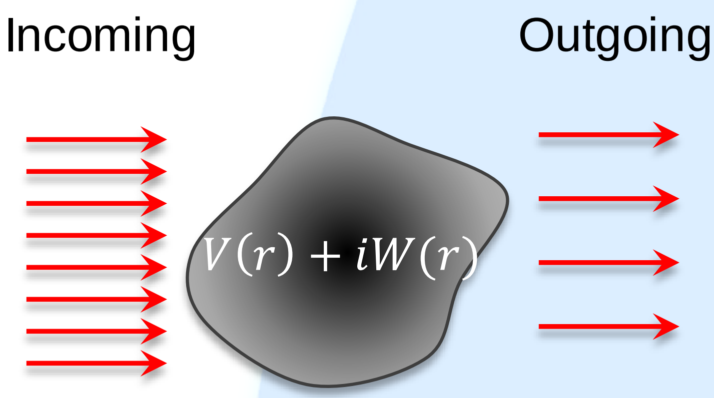


- The inclusion of inelastic channels requires to solve a complex many-body problem.
  Example: We should solve the a+A, C, b+B etc. interacting problem to get their scattering states
- If we are interested in elastic cross-section then inelastic channels happen as a loss of flux
- In some situation, the coupling to inelastic channels can be effectively accounted by introducing an imaginary potential to reduce the flux

$$V(r) \longrightarrow V(r) + iW(r)$$

Optical potential

THE IDEA BEING OPTICAL POTENTIAL



Scattering equation with an imaginary potential

$$\left(\Delta + k^2 - \frac{2\mu}{\hbar^2}\left(V(\vec{r}) + iW(\vec{r})\right)\right)\varphi(\vec{r}) = 0$$

Let's check that some flux is lost:

Current is $\vec{J} = \frac{\hbar}{2\mu i}\left((\varphi)^*\nabla\varphi - \varphi\nabla(\varphi)^*\right)$

Density is $\rho = \varphi^*\varphi$

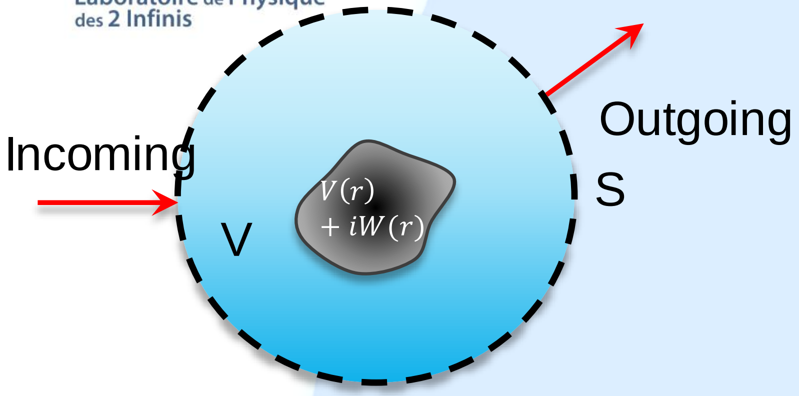
$$\varphi^*(\vec{r}) \times \left(\Delta + k^2 - \frac{2\mu}{\hbar^2}\left(V(\vec{r}) + iW(\vec{r})\right)\right)\varphi(\vec{r}) = 0$$

$$\varphi(\vec{r}) \times \left(\Delta + k^2 - \frac{2\mu}{\hbar^2}\left(V(\vec{r}) - iW(\vec{r})\right)\right)\varphi^*(\vec{r}) = 0$$

$$\hbar\, \vec{\nabla}\cdot\vec{J} = 2W(r)\rho(\mathrm{r})$$

If $W(r) < 0$ Local reduction of the flux

IMAGINARY POTENTIAL: INFLUENCE ON THE CROSS-SECTION



With start with the conservation of matter

$$\frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{J}$$

Which integral form is

$$\frac{d}{dt}\int \rho dV = -\int \vec{\nabla}\cdot\vec{J}dV = \int \vec{J}\cdot\vec{n}dS$$

From the previous result [$\hbar\,\vec{\nabla}\cdot\vec{J} = 2W(r)\rho(\mathrm{r})$] we immediately obtained the lost outgoing flux $-\frac{2}{\hbar}\int d^3rW(r)\rho(\mathrm{r})$, then the absorption cross-section reads

$$\sigma_{\mathrm{abs}} = -\frac{2}{\hbar v}\int d^3rW(r)\rho(\mathrm{r})$$

Which decomposes on partial waves

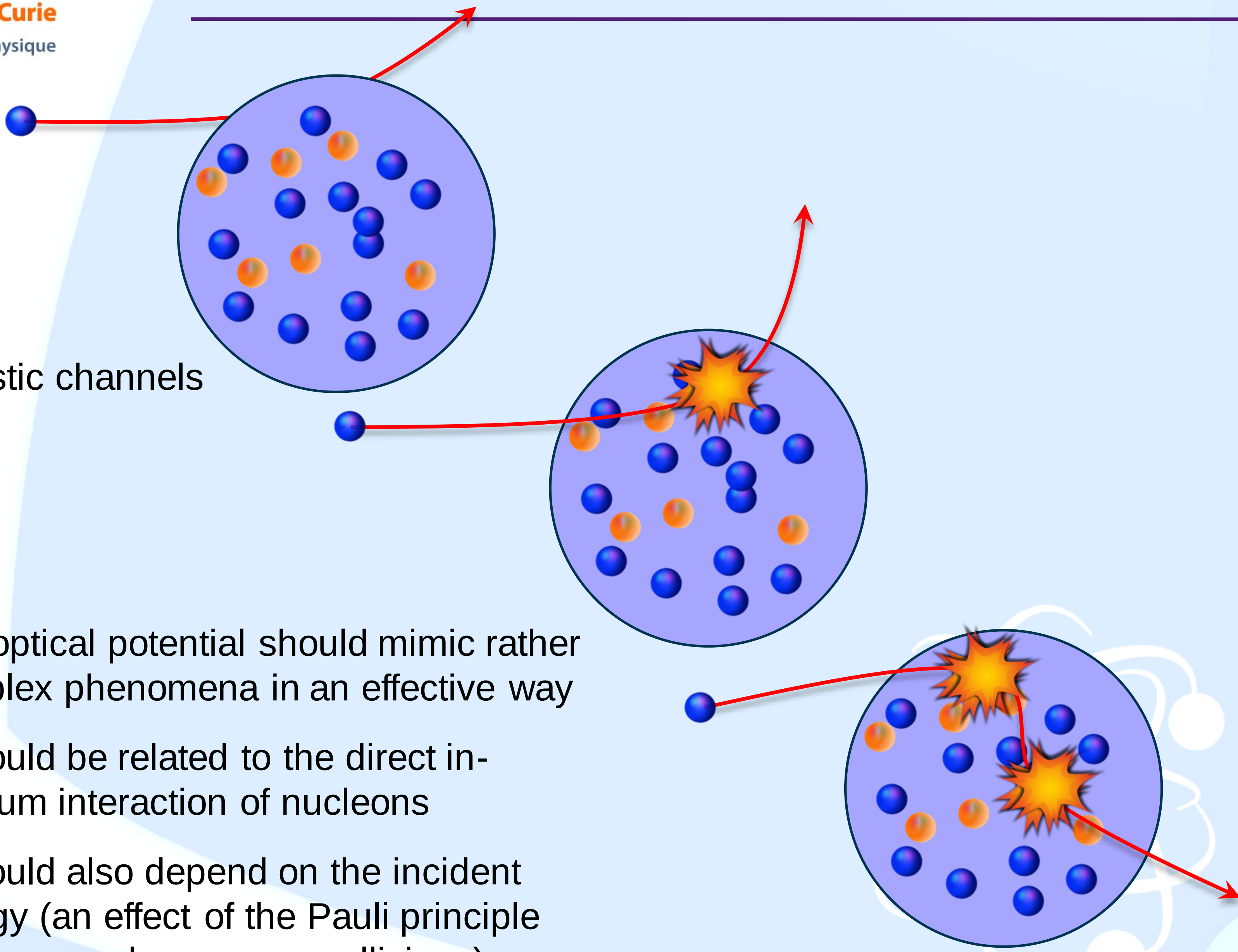
$$\sigma_{\mathrm{abs}} = -\frac{2}{\hbar v}\frac{4\pi}{k^2}\sum(2l+1)\int |u_l(r)|^2W(r)dr$$

To compare with

$$\frac{\pi}{k^2}\sum(2l+1)(1-\left|\tilde{S}_{\alpha,l}\right|^2)$$

$$1-\left|\tilde{S}_{\alpha,l}\right|^2 = -\frac{8}{\hbar v}\int |u_l(r)|^2W(r)dr$$

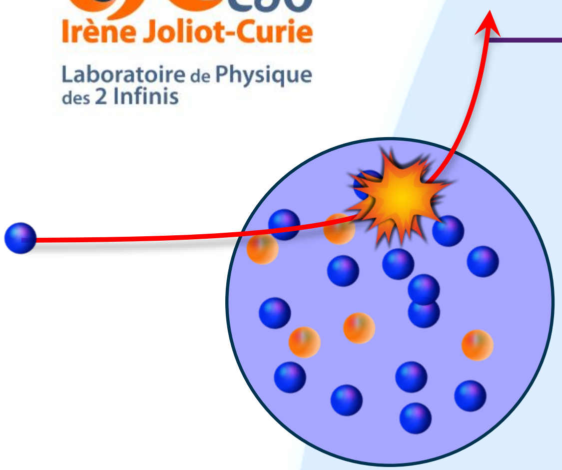
PHYSICAL INTERPRETATION : MEAN-FREE PATH



Non-elastic channels

1. The optical potential should mimic rather complex phenomena in an effective way
2. It should be related to the direct in-medium interaction of nucleons
3. It should also depend on the incident energy (an effect of the Pauli principle that screens low-energy collisions)

PHYSICAL INTERPRETATION : MEAN-FREE PATH



Definition:

Mean-free path: average distance traveled by a nucleon without making collisions with other nucleons

Connection between the optical potential and the mean-free path

Suppose a uniform system with constant potential $V = -(V_0 + iW_0)$, the w.f. reads

$\Psi(\vec{r}) = e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}$

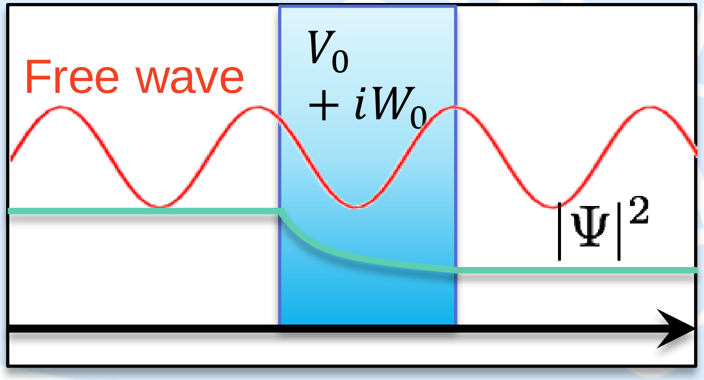
With $\kappa^2 = \frac{2\mu}{\hbar^2}(E + V_0 + iW_0)$ At high energy $W_0 \ll E + V_0$

$$\kappa = \left(\frac{2\mu}{\hbar^2}(E + V_0)\right)^2 \left(1 + \frac{1}{2}\frac{iW_0}{E + V_0}\right)$$

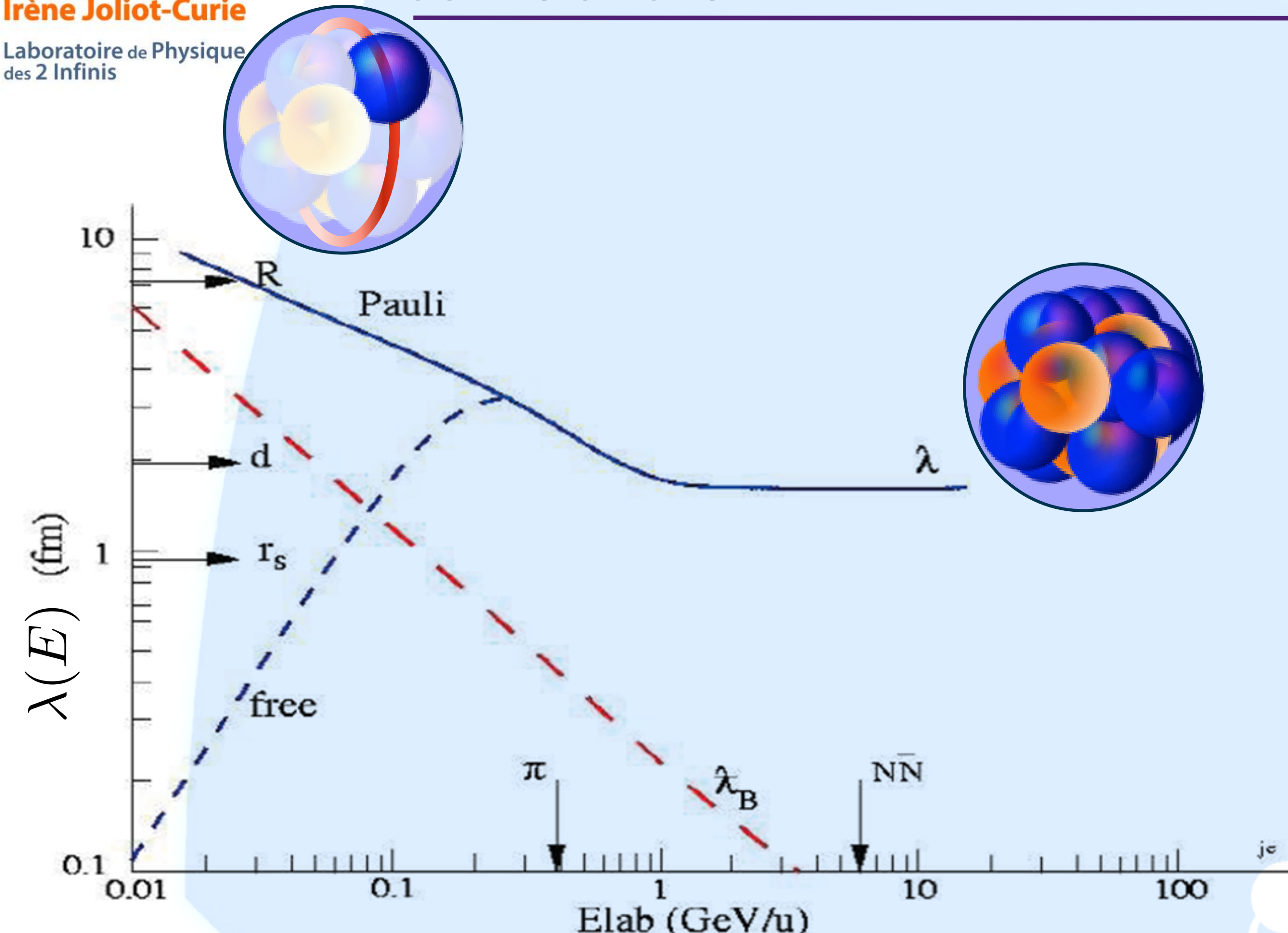
$$\Psi(\vec{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-\frac{r}{\lambda}}$$

With

$$\frac{\hbar^2 k^2}{2\mu} = E + V_0, \qquad \lambda = \frac{\sqrt{\frac{2}{\mu}}\sqrt{E + V_0}}{|W_0|}$$



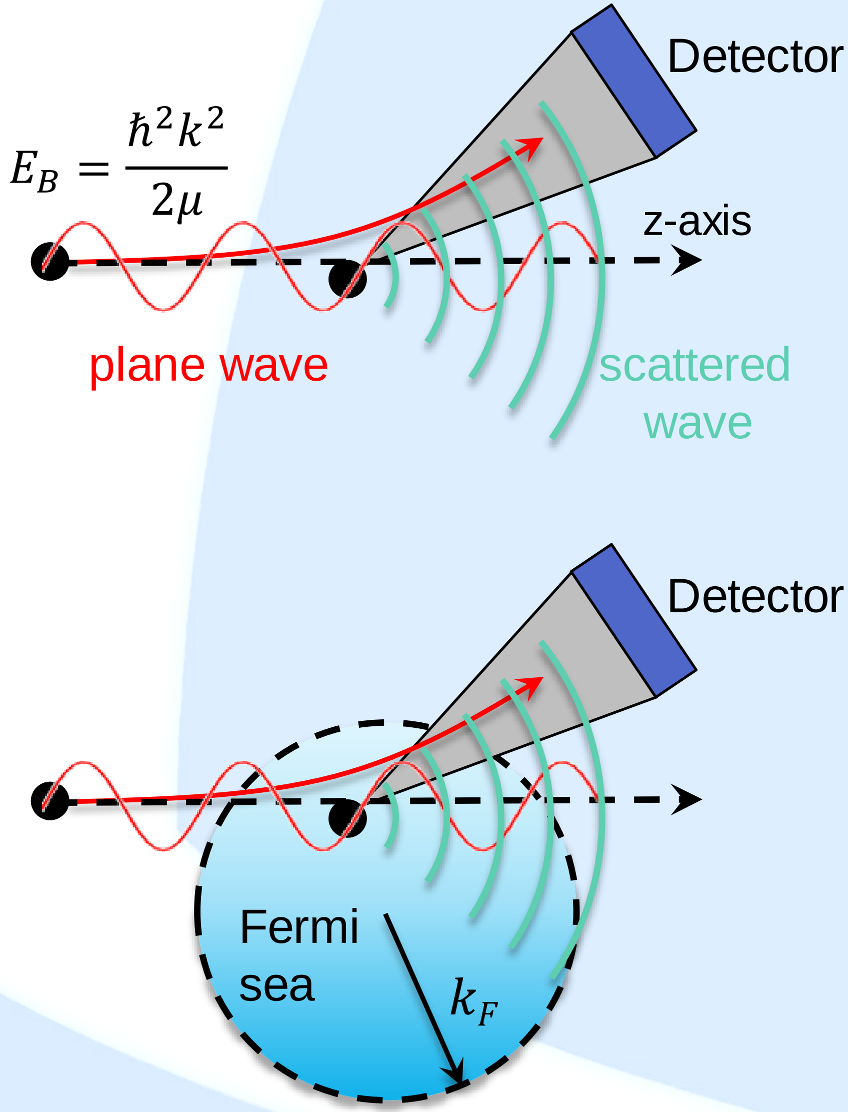
GLOBAL VIEW OF THE MEAN-FREE PATH FROM THE INDEPENDENT PARTICLE PICTURE TO THE DIRECT NN COLLISION CASE



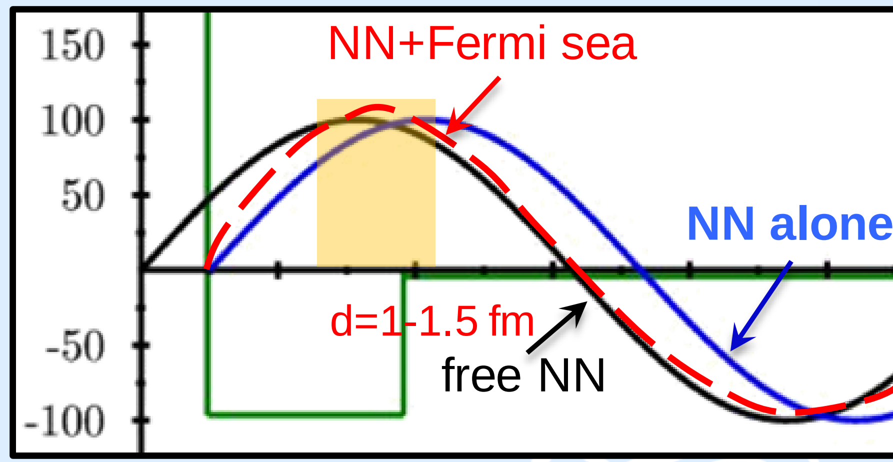
(From J. Cugnon EJC2007)

At very low energy, essentially the average mean-field is felt by the incident nucleon

WHY THE INDEPENDENT PARTICLE WORKS ?



$\Delta k = 0.6\ kF$



Pauli blocking effect strongly “inhibits” in-medium collisions

Gomes, Walecka and Weisskopf, Ann. Phys. (1958)

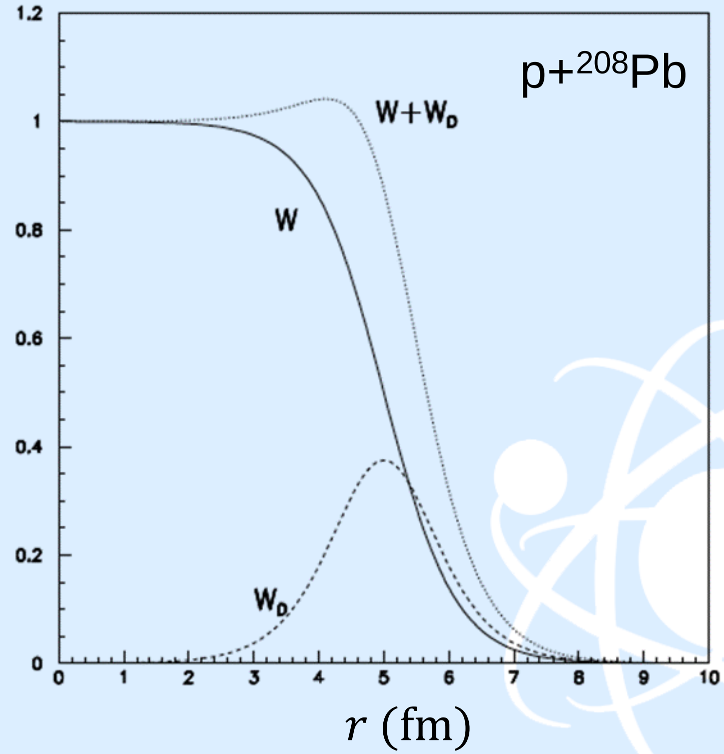
ILLUSTRATION OF OPTICAL POTENTIAL IN NUCLEON-NUCLEUS CASE

Example of Phenomenological optical potential **(From E. Bauge, EJC 2007)**

$$V(r,E) = V(E)f(r,R_v,a_v) + 4V_D(E)f'\left(r,R_{v_D},a_{v_D}\right)$$
$$W(r,E) = W(E)f(r,R_w,a_w) + 4W_D(E)f'\left(r,R_{w_D},a_{w_D}\right)$$

$$f(r,R,a) = \frac{1}{1+e^{\frac{r-R}{a}}}$$
$$f'\left(r,R_{w_D},a_{w_D}\right) = \frac{d}{dr}f(r,R,a)$$

- The parameters are varied until agreement with experiments for total, elastic and absorption cross-section is reached



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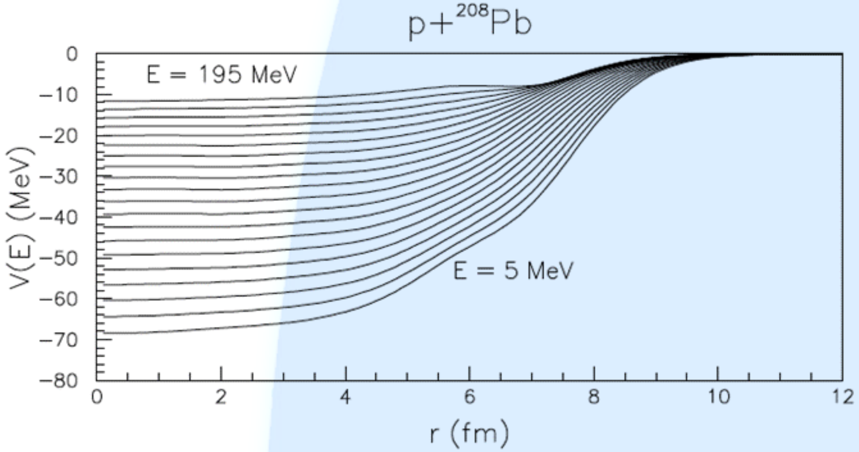
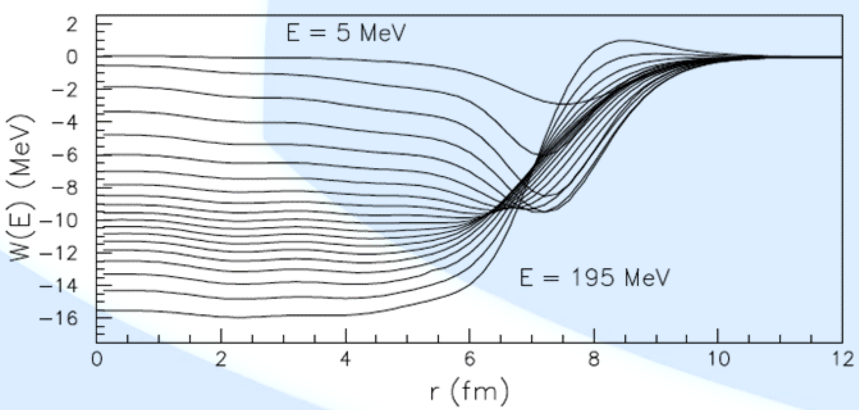
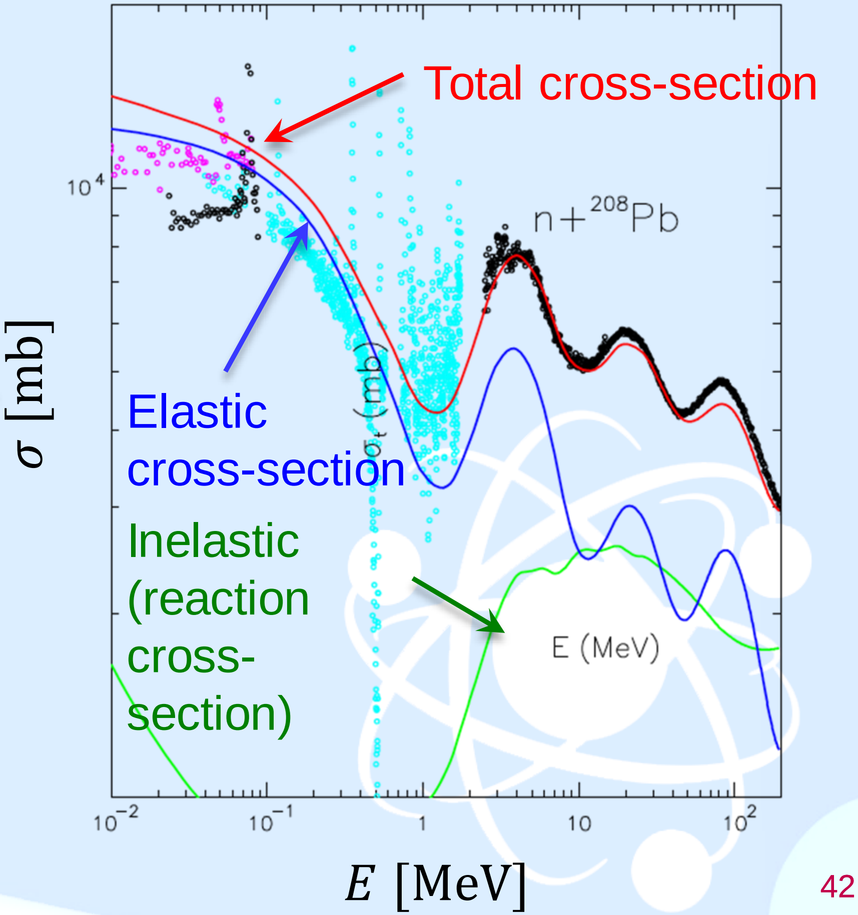
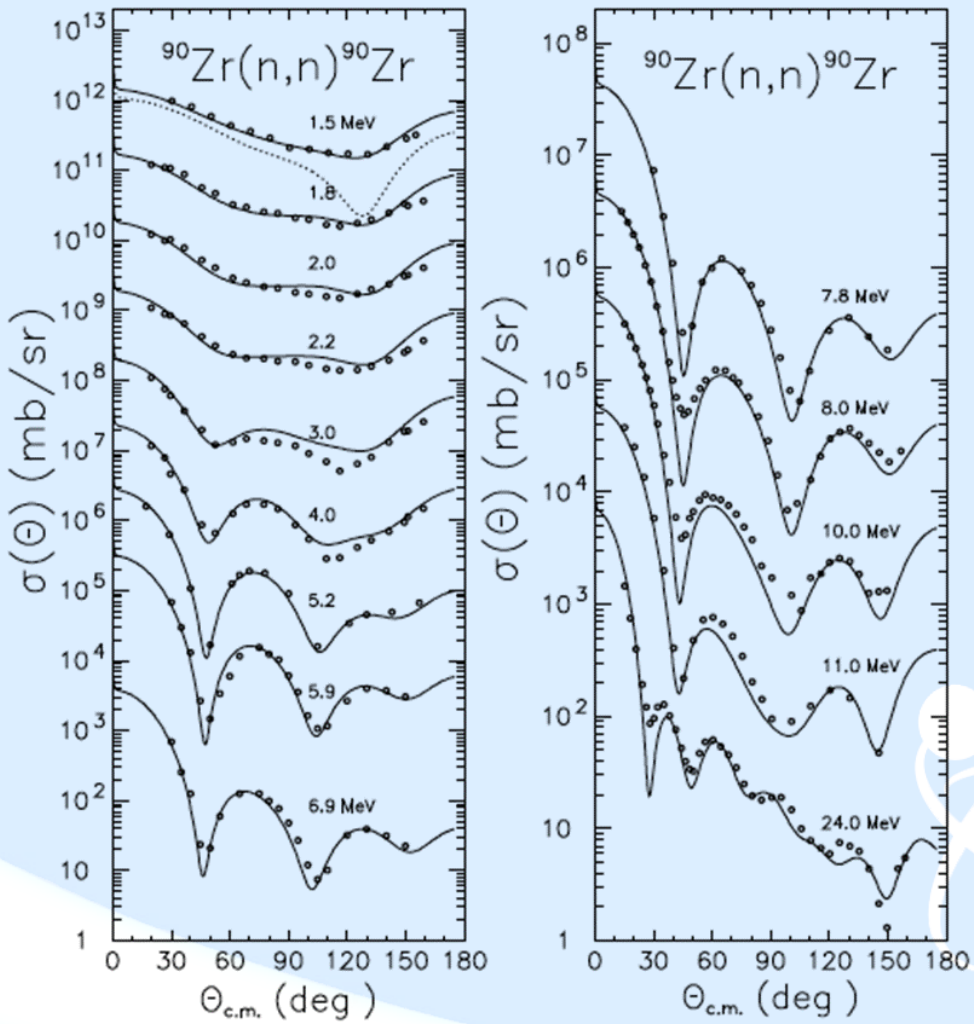
$$W(r,E) = W(E)f(r,R_w,a_w) + 4W_D(E)f'\left(r,R_{w_D},a_{w_D}\right)$$







Differential neutron cross-sections on ^{90}Zr for a beam between 1.5 MeV to 24 MeV



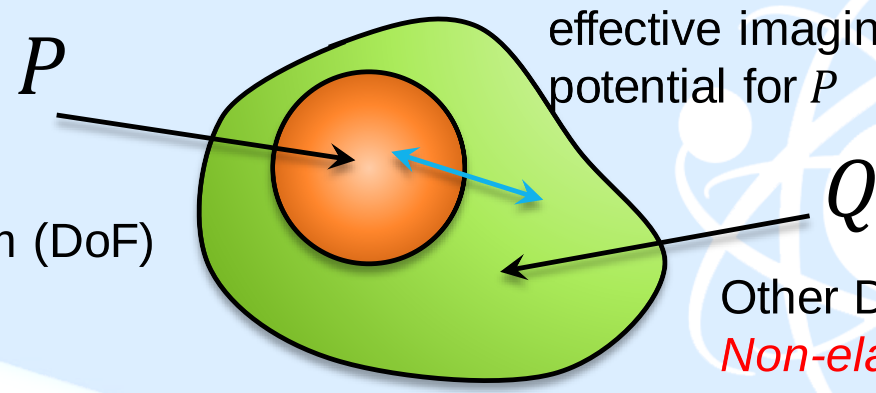
(From E. Bauge, Ecole Joliot-Curie 2007)

EXAMPLE OF DIFFERENTIAL CROSS-SECTION
SOME REMARKS AND CURRENT TREND

- The optical potential is a powerful model to reproduce data
- However, it remains a global fit of the experimental data
- It does in general not tell much about the underlying physical process
- The actual tendency is to provide as much as possible microscopic information on the physical processes leading to non-elastic channels (excitation of target and projectile, direct reactions, ….)
- One standard systematic theory is the Feshbach theory of nuclear reactions + Brückner Hartree-Fock approach (G-matrix)

The coupling induces an effective imaginary potential for P

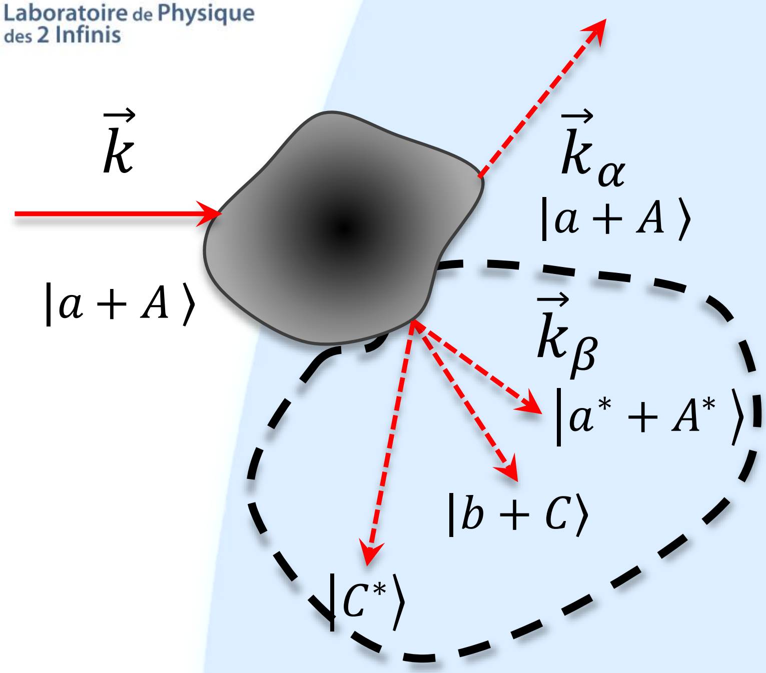
Relevant space of degrees of freedom (DoF)
Elastic channels



Other DOF
Non-elastic channels

(For more details, see E. Bauge Ecole Joliot-Curie 2007)

BACK TO THE SCATTERING PROBLEM WITH NON-ELASTIC CHANNELS



In some cases, it is possible to mimic inelastic channels by an optical potential

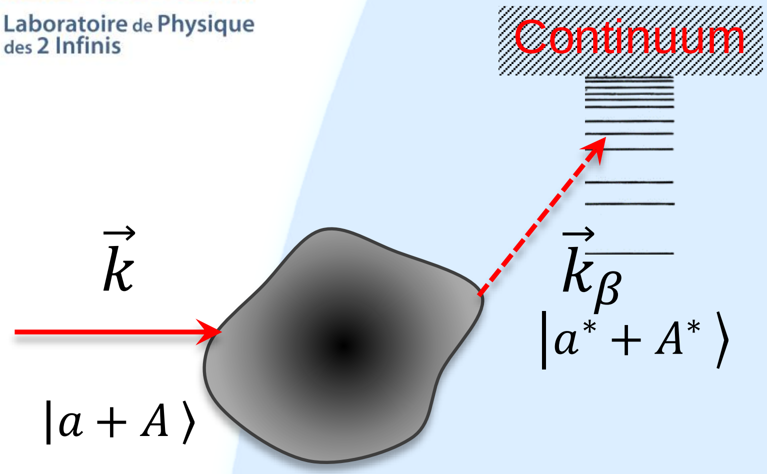
$$1-\left|\tilde{S}_{\alpha,l}\right|^2=-\frac{8}{\hbar v}\int |u_l(r)|^2 W(r)dr$$

In many situation the scattering problem should be directly solved approximately

$$\psi_k^+(\vec{r})\longrightarrow\left(e^{i\vec{k}.\vec{r}}\Psi_a\Psi_A+\sum_\beta \tilde{f}_\beta(\Theta,\varphi)\frac{e^{ik_\beta r}}{r}\Psi_{i_\beta}\Psi_{I_\beta}\right)$$

$$\frac{d\sigma_\beta}{d\Omega}=\frac{v_\beta}{v}\left|f_\beta(\Theta,\varphi)\right|^2=\left|\tilde{f}_\beta(\Theta,\varphi)\right|^2$$

COUPLED CHANNEL METHOD



For $E \sim \mathrm{MeV}$, first inelastic channel is

$$a + A \longrightarrow a^* + A^*$$

In that case both entrance and exit channels are a solution of the same scattering equation that is:

$$H = H_a + H_A - \frac{\hbar^2}{2\mu}\Delta_{\vec{r}}^2 + V(r)$$

Technically, should derive from the same NN+3N+… microscopic interaction

Internal state of $|a\rangle$

Internal state of $|A\rangle$

Relative motion of $|a + A\rangle$

$$H_a \Psi_a^i = E_i^a \Psi_a^i$$

$$H_A \Psi_A^j = E_j^A \Psi_A^j$$

Reaction observables

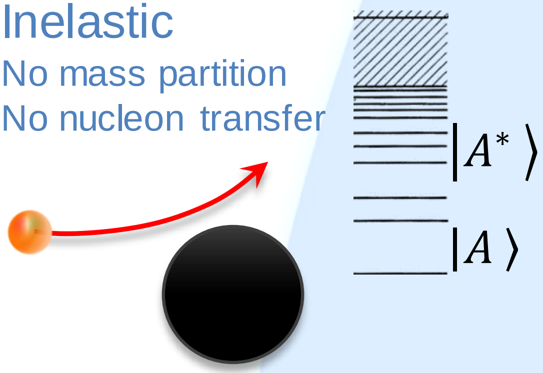
SIMPLIFIED SITUATION

Laboratoire de Physique des 2 Infinis

Inelastic

No mass partition

No nucleon transfer



$$H = H_a + H_A - \frac{\hbar^2}{2\mu_\alpha}\Delta^2_{\vec{r}_\alpha} + V(r_\alpha)$$

The scattering problem can be solved by writing the eigenstates as

$$H\Psi = E\Psi \quad \text{with} \quad \Psi = \sum\nolimits_{x=\{i,j\}} \chi_x(\boldsymbol{r}_\alpha)\Psi_a^i\Psi_A^j$$

$$\sum_x \left[\left(E_a^i + E_A^j - E\right) - \frac{\hbar^2}{2\mu_\alpha}\Delta^2_{\vec{r}_\alpha} + V(r_\alpha)\right]\chi_x(\boldsymbol{r}_\alpha)\Psi_a^i\Psi_A^j = 0$$

$$\left[\nabla_\alpha^2 - U_{x,x}(\mathbf{r}_\alpha) + k_x^2\right]\chi_x(\boldsymbol{r}_\alpha) = \sum_{x'\neq x} U_{x,x'}(\mathbf{r}_\alpha)\,\chi_{x'}(\boldsymbol{r}_\beta)$$

Set of coupled-channel Equation → solve system of linear equation

$$k_x^2 = \frac{2\mu_\alpha}{\hbar^2}\left(E - E_a^i - E_A^j\right)$$

$$U_{x,x'}(\mathbf{r}_\alpha) = \frac{2\mu_\alpha}{\hbar^2}\left\langle \Psi_a^i\Psi_A^j \middle| V_\alpha \middle| \Psi_a^i\Psi_A^j \right\rangle = \frac{2\mu_\alpha}{\hbar^2}\iint \left(\Psi_a^i\right)^*(\boldsymbol{\tau}_a)\left(\Psi_A^j\right)^*(\boldsymbol{\tau}_A)V_\alpha\Psi_a^i(\boldsymbol{\tau}_a)\Psi_A^j(\boldsymbol{\tau}_a)$$

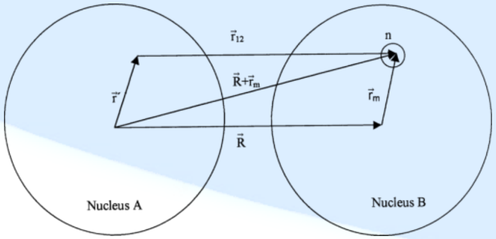
COUPLED-CHANNEL METHOD : SOME REMARKS

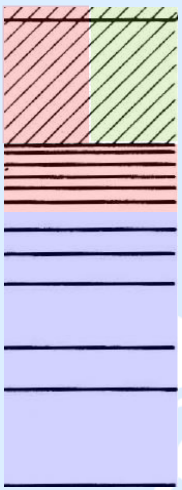
$$\left[\nabla_\alpha^2 - U_{x,x}(r_\alpha) + k_x^2\right]\chi_x(r_\alpha) = \sum_{x' \neq x} \chi_{x'}(r_\alpha) U_{x,x'}(r_\alpha)$$

Diagonal: elastic channels

Off-diagonal: coupling to other channels

- The number of channels is a priori infinite then the method can be combined with optical potential
- Different mass partitions can be included (nucleons transfer) at the price of increasing the number of channel and of computing more terms related to overlaps between mass partitions
- Computation of potential acting on the relative motion is tedious



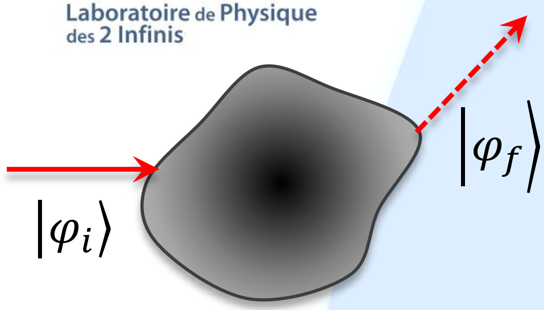


Continuum discretization

Treated by optical potential

Treated explicitly

A SCHEMATIC VIEW OF THE SCATTERING PROCESS



We search for a specific solution of the scattering problem

$$(H - E)|\Psi\rangle = 0$$

Compatible with the known incoming wave function $|\varphi_i\rangle$ (a plane wave). This w.f. ($|\psi_k^+\rangle$) writes

$$|\psi_{\boldsymbol{k}}^+\rangle = \underline{|i_{\vec{k}}\rangle\langle i_{\vec{k}}|\psi_{\boldsymbol{k}}^+\rangle} + \sum |f_{\vec{k}'}\rangle\langle f_{\vec{k}'}|\psi_{\boldsymbol{k}}^+\rangle \qquad \frac{d\sigma_{i\to f}}{d\Omega} \propto \left|\langle f_{\vec{k}'}|\psi_{\boldsymbol{k}}^+\rangle\right|^2$$

$$e^{i\vec{k}\cdot\vec{r}}\Psi_a^0\,\Psi_A^0$$

Elastic	Inelastic	Inelastic
$\Psi_a\Psi_A\chi_k^\beta(\vec{r})$	$\Psi_a^i\Psi_A^j\chi_k^\beta(\vec{r})$	$\Psi_b^i\Psi_B^j\chi_k^\beta(\vec{r})$
$H_a\Psi_a^0 = E_{\mathrm{g.s.}}^a\Psi_a^0$	$H_a\Psi_a^i = E_i^a\Psi_a^i$	$H_b\Psi_b^i = E_i^b\Psi_b^i$
$H_A\Psi_A^0 = E_{\mathrm{g.s.}}^A\Psi_A^0$	$H_A\Psi_A^j = E_j^A\Psi_A^j$	$H_C\Psi_C^j = E_j^C\Psi_C^j$
$\mu_\alpha = \mu_\beta$	$\mu_\alpha = \mu_\beta$	$\mu_\alpha \neq \mu_\beta$
$k = k_\beta$	$k \neq k_\beta$	$k \neq k_\beta$

SCATTERING PROCESS

$$H = H_a + H_A - \frac{\hbar^2}{2\mu_\alpha}\Delta^2_{\vec{r}_\alpha} + V(r_\alpha)$$

α: entrance, elastic

$$|\alpha\rangle = |a + B\rangle$$

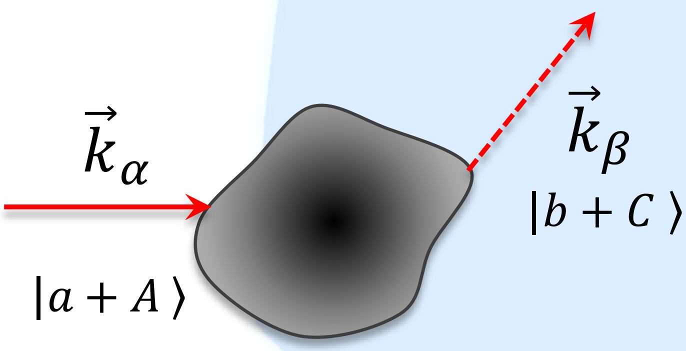
β: exit channels with change of the chemical potential composition

$$H = H_b + H_C - \frac{\hbar^2}{2\mu_\beta}\Delta^2_{\vec{r}_\beta} + V(r_\beta)$$

$$|\beta\rangle = |b + C\rangle$$

We have to deal with non orthonormal and overcomplete basis $\{|\alpha\rangle, |\beta\rangle\}$: $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$

The strategy is just the same taking care of the non-orthogonality



$$|\psi^+_{\boldsymbol{k},\alpha}\rangle = \sum |\varphi_{\boldsymbol{k},\beta}\rangle\langle\varphi_{\boldsymbol{k},\beta}|\psi^+_{\boldsymbol{k},\alpha}\rangle$$

$$|f_{\beta\alpha}|^2 = |f_{\alpha\to\beta}|^2 = |\langle\varphi_{\boldsymbol{k},\beta}|V|\psi^+_{\boldsymbol{k},\alpha}\rangle|^2$$

Probability to populate channel β given the entrance channel

$$\frac{d\sigma_{\beta\alpha}}{d\Omega} = \frac{v_\beta}{v_\alpha}|f_{\beta\alpha}(\Theta,\varphi)|^2$$

$$\frac{d\sigma_{\beta\alpha}}{d\Omega} = \frac{\mu_\alpha\mu_\beta}{(2\pi\hbar^2)^2}\left(\frac{k_\beta}{k_\alpha}\right)|T_{\beta\alpha}(\boldsymbol{k}_\alpha, \boldsymbol{k}_\beta)|^2$$

SCATTERING AMPLITUDE FROM LS EQUATION ETC.

The scattering states are the solution of [prior form]

$$\left[\nabla_\beta^2 + k_\beta^2\right]\chi_\alpha(\boldsymbol{r}_\alpha) = \Omega_\alpha(\boldsymbol{r}_\alpha)$$

As before the solution is formally

$$\left\langle \varphi_{\boldsymbol{k},\beta} \middle| \chi^+_{\boldsymbol{k},\alpha}\right\rangle = \frac{e^{i\vec{k}_\alpha\cdot\vec{r}}}{(2\pi)^{3/2}}\delta_{\alpha,\beta} - \frac{2\mu_\alpha}{\hbar^2}\int d^3 r_\beta{}' \frac{e^{ik_\beta\left|r_\beta - r'_\beta\right|}}{4\pi\left|r_\beta - r'_\beta\right|}\left\langle \varphi_{\boldsymbol{k},\beta}\middle|\Omega_\alpha(\vec{r}'_\beta)\middle|\chi^+_{\boldsymbol{k},\alpha}\right\rangle$$

$$= T_{\beta\alpha}$$

This is the Lippmann-Schwinger equation

Born approximation

$$\chi^+_{\boldsymbol{k},\alpha}(\boldsymbol{r}_\alpha) = \varphi_{\boldsymbol{k},\boldsymbol{\alpha}}(\boldsymbol{r}_\alpha) = \frac{e^{i\vec{k}_\alpha\cdot\vec{r}_\alpha}}{(2\pi)^{3/2}}$$

Distorted Wave approximation

$$\left[\nabla_\beta^2 - U_{\alpha,\alpha}(\mathbf{r}_\alpha) + k_\beta^2\right]\chi_\alpha(\boldsymbol{r}_\alpha) = \Omega_\alpha(\boldsymbol{r}_\alpha)$$
$$\left[\nabla_\beta^2 - U_{\alpha,\alpha}(\mathbf{r}_\alpha) + k_\beta^2\right]\chi^-_\beta(\boldsymbol{r}_\alpha) = 0$$

$$T_{\beta\alpha} = \left\langle \chi^-_{\boldsymbol{k},\beta}\middle|\Omega_\alpha(\vec{r}'_\beta)\middle|\chi^+_{\boldsymbol{k},\alpha}\right\rangle$$

TO GO MESSAGE

- The theory of scattering by a general potential is rather cumbersome with many degrees of sophistication
- But it is used in many area of physics
- Normally particles have internal DoF, are often fermions and have spin/isospin that recouple with angular momentum/total isospin.
- This makes the theory of scattering even more technical
- Without using it we forget almost as fast as we learn this theory
- Please remember more the general strategy/physical meaning than the technical details

$$|\Psi^+_{\rm in}\rangle = |{\rm in}\rangle\langle{\rm in}|\Psi^+_{\rm in}\rangle + \sum|{\rm out}\rangle\langle{\rm out}|\Psi^+_{\rm in}\rangle$$