NPAC Accelerator Physics Examination

Longitudinal dynamics and space charge

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Muon colliders have a great potential for high-energy physics. They can offer collisions of point-like particles at very high energies, since muons can be accelerated in a ring without limitation from synchrotron radiation. However, the need for high luminosity faces technical challenges which arise from the short muon lifetime at rest and the difficulty of producing large numbers of muons in bunches with small emittance. Addressing these challenges requires the development of innovative concepts and demanding technologies.

A few useful physical constants and formulae

For the document, we will use the following notations and values for the constant:

- Speed of light: $c = 3 \times 10^8\,\mathrm{m/s}$
- Particle velocity: v
- Reduced velocity: $\beta = \frac{v}{c}$
- Lorentz factor: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$
- Muon rest mass: $m_\mu c^2 = 100\,\mathrm{MeV}$
- Muon lifetime: $\tau_\mu = 2.2\,\mu\mathrm{s}$
- Total momentum: $p = \beta\gamma c$
- Charge number: q
- Absolute phase: φ
- Total energy: $E = \gamma m_\mu c^2$
- Index for the synchronous particle : s
- Synchronous phase: φ_s
- Relative phase difference: $\phi = \varphi - \varphi_s$
- Relative energy difference: $\delta E = E - E_s$
- Magnetic rigidity: $B\rho = \frac{p}{q}$
- Dipole angle: $\theta = \frac{\int B(s)ds}{B\rho} \approx \frac{BL}{B\rho}$
- Total RF voltage: $V_0 = \int |E(s)|ds$
- Transit time factor: $T(\beta) = \frac{\left|\int E(s) \exp \imath \frac{\omega s}{\beta c} ds\right|}{V_0}$
- Momentum compaction: α_p
- Slip factor: $\eta = \frac{1}{\gamma^2} - \alpha_p$
- Hamiltonian in longitudinal plane:

$$H(\phi, \delta E; s) = \frac{\pi \cdot \eta}{\lambda_{\mathrm{RF}}} \frac{\delta E^2}{\beta_s^3 \gamma_s m c^2} - qE_0T\left(\sin\varphi_s\left(\phi - \sin\phi\right) - \cos\varphi_s\left(1 - \cos\phi\right)\right)$$

- Number of muons: $\frac{dN(t)}{dt} = -\frac{N(t)}{\gamma(t)\tau_\mu}$

Introduction

A schematic layout of Muon Collider complex based is sketched below.

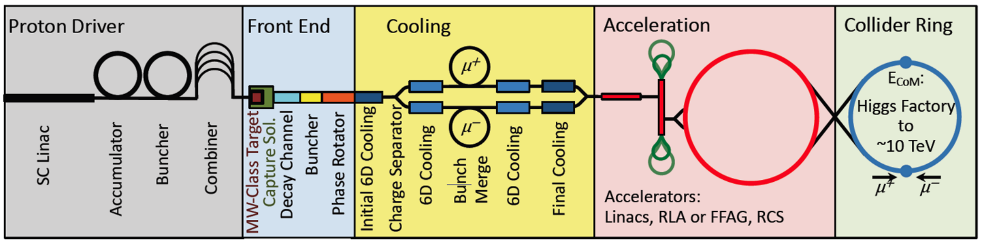

Figure 1: Proton complex for a muon collider complex

The functional elements of the muon beam generation and acceleration systems are:

- a proton driver producing a high-power multi-GeV, multi-MW bunched H beam,
- a buncher made of an accumulator and a compressor that forms intense and short proton bunches,
- a pion production target in a heavily shielded enclosure able to withstand the high proton beam power, which is inserted in a high field solenoid to capture the pions and guide them into a decay channel,
- a front-end made of a solenoid decay channel equipped with RF cavities that captures the muons longitudinally into a bunch train, and then applies a time-dependent acceleration that increases the energy of the slower (low-energy) bunches and decreases the energy of the faster (high-energy) bunches,
- an "initial" cooling channel that uses a moderate amount of ionization cooling to reduce the 6D phase space occupied by the beam by a factor of 50 (5 in each transverse plane and 2 in the longitudinal plane), so that it fits within the acceptance of the first acceleration stage. For high luminosity collider applications, further ionization cooling stages are necessary to reduce the 6D phase space occupied by the beam by up to five orders of magnitude,
- the beam is then accelerated by a series of fast acceleration stages such as Recirculating Linacs Accelerators (RLA) and Rapid Cycling Synchrotron (RCS) to take the muon beams to the relevant energy before injection in the muon collider Ring.

Part 1: Muon cooling

The aim of the muon cooling channel is to reduce the beam emittance based on the ionization cooling. The kinetic energy of the muons in this channel is 200 MeV. The total length is 400 m. We assume that the muon energy is constant in the channel.

In ionization cooling, muons pass through a material medium and lose energy (momentum) through ionization interactions, and this is followed by beam reacceleration in RF cavities.

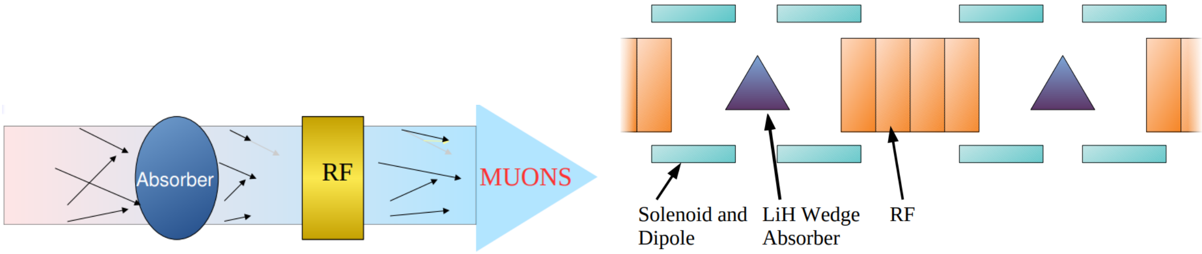

Figure 2: Principle (left) and cell proposal (right) of the ionization cooling.

1. Calculate the value of γ and β of the muons.

2. With a simple scheme, show the momentum variation while crossing an ionization cell to explain why we can get a reduction of the transverse emittance.

3. When the beam crosses the target, we have also an emittance growth due to the Coulomb scattering. The evolution of the normalized transverse emittance and of the energy spread can be given by:

$$
\begin{aligned}
\frac{d\epsilon_N}{ds} &= -\frac{1}{\beta^2 E}\frac{dE}{ds}\epsilon_N + \frac{\beta_\perp E_{\text{scat}}^2}{2\beta^2 m_\mu c^2 L_R E} \\
\frac{d\sigma_E^2}{ds} &= -2\frac{\partial \frac{dE}{ds}}{\partial E}\sigma_E^2 + \frac{d\left\langle \Delta E_{\text{rms}}^2\right\rangle}{ds} \\
\frac{dE}{ds} &= 4\pi N_A r_e^2 m_e c^2 \rho \frac{Z}{A}\left[\frac{1}{\beta^2}\log\left(\frac{2m_e c^2\gamma^2\beta^2}{16Z^{0.9}}\right) - 1\right] \\
\frac{d\left\langle \Delta E_{\text{rms}}^2\right\rangle}{ds} &= 4\pi\left(r_e m_e c^2\right)^2 n_e \gamma^2\left(1 - \frac{\beta^2}{2}\right)
\end{aligned}
$$

where $\frac{dE}{ds}$ is the energy loss rate in the material, $\frac{d\left\langle \Delta E_{\text{rms}}^2\right\rangle}{ds}$ is a heating term because of random fluctuation in the energy loss, L_R the material radiation length, ρ the target density, $r_e$the classical electron radius, $\beta_\perp$ the β-function at the absorber, $E_{\text{scar}} \approx 13.6\,\text{MeV}$.

   a) Give the expression of the transverse emittance at equilibrium.

   b) Express $\beta_\perp$ as a function of the solenoid field and beam energy by assuming that the beam radius is constant in the solenoid and with negligible space-charge forces. We assume to be at the equilibrium, the envelope equation is:

$$
\tilde{x} = \sqrt{\beta_x \epsilon_x} \qquad\qquad \tilde{x}'' + k_{x,0}^2(s)\cdot\tilde{x} - \frac{K/2}{\tilde{x}+\tilde{y}} - \frac{\epsilon_x^2}{\tilde{x}^3} = 0
$$

   The focusing strength in a solenoid gives $k_{x,0} = \frac{B}{2B\rho}$. We assume that the beam is round $\tilde{x} = \tilde{y}$ and that space-charge effects are negligible (perveance $K = 0$).

c) To go to smaller transverse emittances, how should the solenoid field change? Do you see the interest of staging the solenoid field during the cooling process?

4. To speed-up the cooling of the longitudinal emittance, an idea is to add some dispersion D_x at the absorber position (the dispersion gives a correlation between the particle position and the energy: $< x >= \frac{\delta E}{E_s} \cdot D_x$) and to use an absorber with a wedge-shape instead of rectangular shape. With no calculation, could you explain how this disposal can change the cooling speed for the longitudinal plane?

Part 2: Muon linac

After cooling the muon beam, the muons μ^+ and μ^- are accelerated in a linac. We will assume that the beam is ultra-relativistic with $\beta = 1$.

5. Give 2 reasons why we cannot use an electrostatic accelerator.

6. Remind in a few words how an RF cavity works and what are the criteria to choose one cavity type instead of another (for instance an elliptical cavity instead of a drift tube).

7. You have the choice between these 3 cavities of pulsation $\omega = 2\pi f$. Which no calculation, which one do you think that is the most relevant?

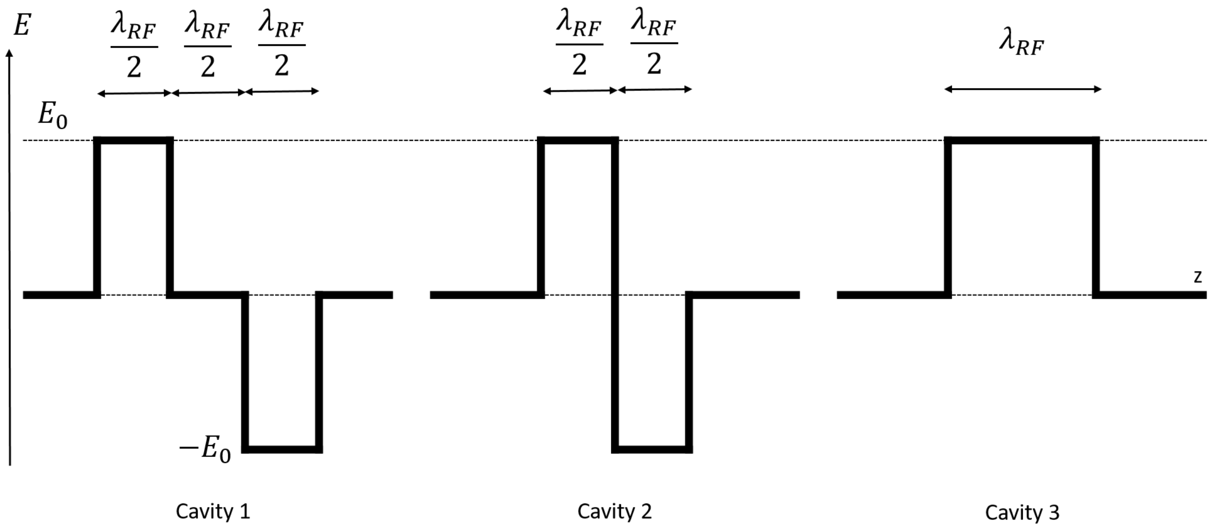

Figure 3: Variation of the field E in the RF cavities along the axis s. λ_{RF} is the RF wavelength

8. Calculate the maximum energy gain for the cavity you have chosen above.

9. How does the transverse space charge force vary with energy? Explain quickly why.

10. We assume that the external force is continuous and purely linear. Show on a sketch how the horizontal motion spectrum varies for a beam with no space charge, for a uniform beam and for a Gaussian beam.

11. The linac is now made of a series of RF cavities. The distance between the RF cavity i and the RF cavity $i+1$ is L_i. We assume that all RF cavities have the same voltage V. We assume that when the synchronous particle enters the first cavity, the phase in the cavity i is $\varphi_{0,i}$. Give the condition on $\varphi_{0,i}$ to keep the same energy gain in each RF cavity.

12. We consider a beam with a uniform transverse distribution of radius a in a beam pipe of radius b. The transverse distribution is then:

$$n(r,z) = Ne\lambda(s-\beta ct)\begin{cases} \dfrac{1}{\pi a^2} & \text{if } r < a \\ 0 & \text{if } r > a \end{cases} \tag{1}$$

We assume that the beam is ultra-relativistic. In this case, the transverse electric field E_r and magnetic field B_θ is determined by the *local* longitudinal beam distribution. In other terms, the electric field can be expressed as:

$$E_r = \frac{B_\theta}{\beta c} = Ne\frac{\lambda(s-\beta ct)}{2\pi\epsilon_0}\begin{cases} \dfrac{r}{a^2} & \text{if } r < a \\ \dfrac{1}{r} & \text{if } r > a \end{cases}$$

By applying Faraday's law $\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A}$ on a rectangle with the summits at (r,s), $(r,s+ds)$, $(b,s+ds)$, and (b,s), show that the longitudinal field is given by:

$$E_z = -\frac{Ne}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\begin{cases} \log\dfrac{b}{a} + \dfrac{1}{2} - \dfrac{r^2}{2a^2} & \text{if } r < a \\ \log\dfrac{b}{r} & \text{if } r > a \end{cases}$$

13. We assume that the longitudinal distribution is parabolic: $\lambda(z) = \frac{3}{4\sigma_z}\left(1-\left(\frac{z}{\sigma_z}\right)^2\right)$.

    We assume here that γ_s is roughly constant. For small amplitudes, the evolution in the longitudinal phase plane is:

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{2\pi\cdot\eta}{\lambda_{\mathrm{RF}}}\cdot\dfrac{\delta E}{\beta_s^3\gamma_s m_\mu c^2} \\ \dfrac{d\delta E}{ds} = qE_0T\cos\varphi_s\phi + qE_z \end{cases}$$

    a) Give the differential equation of ϕ.

b) Give the tune with no space charge and the tune depression for an on-axis particle.

c) Is the longitudinal space charge defocusing or defocusing for a linac and a high-energy synchrotron? Compare to the transverse case.

d) Do you see a way to keep the same energy gain and tune?

Part 3: The RCS

We consider now the first rapid cycling synchrotron after the linac. The injection energy is $E_0 = \gamma_0 m_\mu c^2 = 60\,\text{GeV}$ and the total circumference is $L_{\text{RCS}} = 6000\,\text{m}$. The maximum dipole field is $1.5\,\text{T}$. The extraction energy is $E_1 = \gamma_1 m_\mu c^2 = 300\,\text{GeV}$. We assume that the RF frequency is $1.3\,\text{GHz}$. We assume that the transition gamma is $\gamma_T = 16$ and an acceleration time t_{acc} of 17 turns. The synchronous phase in the RF cavities is $\varphi_s = 135°$ (We use the sine convention.). In this part, we assume that the bunch length is small.

14. Calculate the harmonic number and the total dipole length.

15. Calculate the momentum compaction. Remind its definition.

16. We assume that we can do the continuous channel approximation. Show in the longitudinal phase space the separatrix, the bucket, the bunch and in which direction the particles are moving.

17. We assume that the ramp is linear: $\gamma(t) = \gamma_0 + \frac{\gamma_1-\gamma_0}{t_{\text{acc}}}t$. Show that the survival rate of the muons is:

$$\frac{N_1}{N_0} = \exp\left[-\frac{1}{\tau_\mu}\frac{t_{\text{acc}}}{\gamma_1-\gamma_0}\log\frac{\gamma_1}{\gamma_0}\right]$$

    Make the numerical application.

18. Give the required total voltage.

19. We inject a beam with a relative energy difference of 10^{-3}. Give the matching condition on the bunch length.

20. Calculate the synchrotron tune at injection $Q_s = k_{s,0}\frac{L_{\text{RCS}}}{2\pi}$. The synchrotron tune is the ratio between the longitudinal period and the revolution period. Do you think that we can do the continuous channel approximation if we locate all RF sections in a straight section?