Particle Accelerators 6

Introduction to collective effects: space-charge

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1. Space-charge force

Generalities on fields: static model

Field produced by charge and current densities

Maxwell equations

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \cdot \mathbf{j} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

These charge ρ and current $\mathbf{j}$ densities are:

- those of the beam (direct space-charge),
- those induced in surrounding material (indirect space-charge).

Two solutions

- Simplified model: static
- Numerical resolution

Static model: application case

Continuous beam

One assumes that charge and current distributions at a given position are **stationary**. Fields are then invariant with time and electric and magnetic fields are independent.

$$\rho(\mathbf{r},t),\mathbf{j}(\mathbf{r},t) \longrightarrow \rho(\mathbf{r}),\mathbf{j}(\mathbf{r})$$

Bunched beam frame

In the beam frame, the particle relative displacements are generally non-relativistic and field is mainly electrostatic.

$$\mathbf{E}^*(\mathbf{r},t),\mathbf{B}^*(\mathbf{r},t) \xrightarrow{\beta^*\ll 1} \mathbf{E}^*(\mathbf{r}),\mathbf{0} \xrightarrow[\text{transform}]{\text{Lorentz}} \mathbf{E}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)$$

Except in specific cases, the magnetic field is not directly computed but **the magnetic force is deduced from the electric force**.

Electrostatic field

Charge distribution

A still **charge density** ρ [C m^{-3}] produces an **electrostatic field**:

$\mathbf{E}$ [V m^{-1}] solution of equations : $\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} = \mathbf{0} \end{cases}$

The solution of two coupled equations is not obvious as once we found a solution of the first, it has to satisfy the second one.

It is then easier to solve a unique equation by remarking that $\nabla \times (\nabla f) = \mathbf{0}$, whatever f.

Defining: $\mathbf{E} = -\nabla\phi$ With ϕ [V] the scalar **electrostatic potential**.
The system becomes:

$$\nabla \cdot (\nabla\phi) = \Delta\phi = -\frac{\rho}{\epsilon_0}$$

Magnetostatic field

Current distribution

A **current flux j** [A m^{-2}] produces a **magnetic field**:

$\mathbf{B}$ [T] solution of equations : $\begin{cases} \nabla \times \mathbf{B} = \mu_0 \cdot \mathbf{j} \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$

The solution of two coupled equations is not obvious as once we found a solution of the first, it has to satisfy the second one.

It is then easier to solve a unique equation by remarking that $\nabla \cdot (\nabla \times \mathbf{f}) = \mathbf{0}$, whatever $\mathbf{f}$.

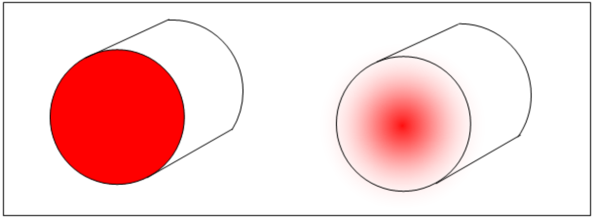
Defining: $\mathbf{B} = \nabla \times \mathbf{A}$ With $\mathbf{A}$ [T m] the **magnetic vector potential**.
The system becomes:

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$$

1. Space-charge force

Continuous beam

Cylindrical continuous beam



$$\rho(x,y,z) \to \rho(r)$$
$$\mathbf{j}(x,y,z) \to j(r)\mathbf{e}_z$$
$$r = \sqrt{x^2+y^2}$$

Gauss theorem: $\oiint \mathbf{E}\cdot\mathbf{dS} = \frac{\iiint \rho d\tau}{\epsilon_0}$

$$\boxed{E_r(r) = \frac{1}{\epsilon_0 \cdot r}\int_0^r r'\cdot\rho(r')\cdot dr'}$$

Ampere theorem: $\oint \mathbf{B}\times\mathbf{dl} = \mu_0 \iiint \mathbf{j} dS$

$$\boxed{B_\theta(r) = \frac{\mu_0}{r}\int_0^r r'\cdot\mathbf{j}(r')\cdot dr'}$$

Cylindrical continuous beams

Some examples

Charge per linear meter:

$$\rho(r) = \frac{\lambda_0}{2\pi r} \cdot \delta(r) \qquad \longrightarrow \qquad E_r(r) = \frac{\lambda_0}{2\pi\epsilon_0} \cdot \frac{1}{r}$$

Uniform beam: electric field linear with r in beam

$$\rho(r) = \begin{cases} \frac{\lambda}{\pi \cdot R_h^2} & \text{if } r < R_h \\ 0 & \text{otherwise} \end{cases} \qquad \longrightarrow \qquad E_r(r) = \begin{cases} \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{r}{R_h^2} & \text{if } r < R_h \\ \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r} & \text{otherwise} \end{cases}$$

Gaussian beam:

$$\rho(r) = \frac{\lambda}{2\pi\sigma_r^2} \cdot \exp\left(-\frac{r^2}{2\sigma_r^2}\right) \qquad \longrightarrow \qquad E_r(r) = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r} \cdot \left(1 - \exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right)$$

- λ [$\mathrm{C\,m^{-1}}$] is the charge per linear meter: $\lambda = \frac{I}{\bar{\beta} \cdot c}$
- I [A] is the beam current
- $\bar{\beta}$ is the beam particle average velocity

Cylindrical continuous beam

electric – magnetic forces

Assuming that all particles have the same velocity: $\mathbf{v} = \bar{\beta}_z c \cdot \mathbf{u}_z$

$$\mathbf{j}(r) = \rho(r) \cdot \bar{\beta}_z c \cdot \mathbf{u}_z \quad \longrightarrow \quad B_\theta(r) = \frac{\mu_0 \cdot c}{r} \cdot \bar{\beta}_z \cdot \int_0^r r' \cdot \rho(r') \cdot dr' = E_r(r) \cdot \frac{\bar{\beta}_z}{c}$$

The force on each particle with charge q and longitudinal **reduced velocity** β_z is:

$$F_r = q\left(E_r - v_z \cdot B_\theta + v_\theta \cdot B_z\right) = q\left(E_r - \beta_z c \cdot B_\theta\right)$$
$$F_r = q \cdot E_r(r) \cdot \left(1 - \beta_z \cdot \bar{\beta}_z\right)$$

Paraxial approximation: $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 \approx \beta_z^2$

$$\boxed{F_r = q \cdot E_r(r) \cdot \left(1 - \beta^2\right) = \frac{q \cdot E_r(r)}{\gamma^2}}$$

F_r scales with $1/\gamma^2$: Laplace force mitigates Coulomb repulsion.

Elliptical uniform continuous beam

$$\rho(x,y,z) = \begin{cases} \frac{\lambda}{\pi \cdot X \cdot Y} & \text{if } \frac{x^2}{X^2} + \frac{y^2}{Y^2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} E_x(x,y) = \dfrac{\lambda}{4\pi^2 \cdot \epsilon_0 \cdot X \cdot Y} \displaystyle\int_{-Y}^{Y} dy' \cdot \int_{-X\sqrt{1-y^2/Y^2}}^{X\sqrt{1-y^2/Y^2}} dx' \frac{x-x'}{\left((x-x')^2 + (y-y')^2\right)^{3/2}} \\ E_y(x,y) = \dfrac{\lambda}{4\pi^2 \cdot \epsilon_0 \cdot X \cdot Y} \displaystyle\int_{-X}^{X} dx' \cdot \int_{-Y\sqrt{1-x^2/X^2}}^{Y\sqrt{1-x^2/X^2}} dy' \frac{y-y'}{\left((x-x')^2 + (y-y')^2\right)^{3/2}} \end{cases}$$

$$\boxed{\begin{cases} E_x(x,y) = \dfrac{\lambda}{\pi \cdot \epsilon_0} \cdot \dfrac{1}{X+Y} \cdot \dfrac{x}{X} \\ E_y(x,y) = \dfrac{\lambda}{\pi \cdot \epsilon_0} \cdot \dfrac{1}{X+Y} \cdot \dfrac{y}{Y} \end{cases}}$$

(In beam)

Electric field linear with position

1. Space-charge force

Numerical methods

Numerical methods

The space-charge field produced by a set of N particles can be calculated with different **space-charge routines**:

- **PPI** (Particle-Particle Interactions) methods
- **PIC** (Particles in cells) methods
  - direct,
  - FFT,
  - relaxation.
- Functional methods

PPI (Particle-Particle Iteraction) Method

At each time step, the field contribution of all particles is calculated at the position of each particle:

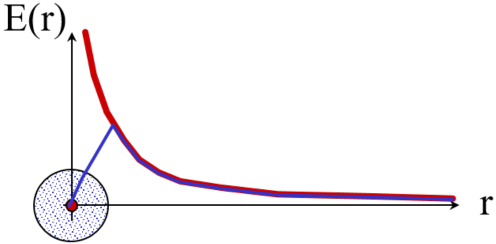
$$\mathbf{E}(\mathbf{r}_i) = \sum_{j \neq i}^{N} \frac{q_j}{4\pi\epsilon_0} \frac{\mathbf{r}_i - \mathbf{r}_j}{\left\|\mathbf{r}_i - \mathbf{r}_j\right\|^3}$$

Advantages

- No mesh
- Easy to compute

Drawbacks

- Long ($\propto N^2$),
- Artificially colliding



PIC (Particles in Cells) method

- Particles are counted in a mesh with n lattices
- In the direct method, the influence of the density in each lattice is calculated on each mesh node.

Advantages

- Low noise (charge smoothing on the mesh)

Drawbacks

- Long ($\propto n^2$),
- No image charge.

PIC FFT method

$$\phi(x_0, y_0, z_0) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{space}} \frac{\rho(x,y,z)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} dx \cdot dy \cdot dz$$
$$= (\rho * G)(x,y,z)$$
$$\text{With :} G = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Then: $\phi(x,y,z) = FFT^{-1}(FFT(\rho) \times FFT(G))$

Advantages

- Fast ($\propto n \cdot \log(n)$)

Drawbacks

- Noisy.
- No image charge.

PIC relaxation method

Illustration in 1D : $\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho(x)}{\epsilon_0} = \rho'(x)$

On each lattice: $\phi_{i+1} - 2\phi_i + \phi_{i-1} = \rho'_i \cdot \delta^2$

Iterative process k: $\phi_i^{k+1} = \phi_i^k + \alpha\left(\frac{\phi_{i+1}^k + \phi_{i-1}^k - \rho'_i \cdot \delta^2}{2} - \phi_i^k\right)$

Advantages

- Could be fast ($\propto n \cdot \log(n)$, for multigrid)
- Image charge

Drawbacks

- Limit condition should be defined (or assumed).

Functional method

$$\rho(\mathbf{r}) = \sum_j A_j \cdot P_j(\mathbf{r}) \qquad \text{with:} \qquad A_j = \sum_{i=1}^{N} F\left(\rho\left(\mathbf{r}_i\right), P_j\left(\mathbf{r}_i\right)\right)$$

with P_j such as: $\Delta\Gamma_j(\mathbf{r}) = P_j(\mathbf{r})$

$$\Rightarrow \quad \phi(\mathbf{r}) = \sum_j A_j \cdot \Gamma_j(\mathbf{r})$$

Advantages

- Mathematically smart.

Drawbacks

- Very long,
- Noisy,
- No image charge

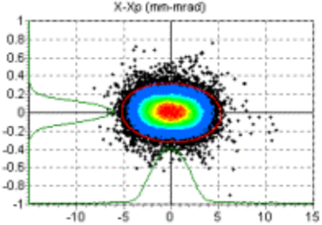
2. Linear(ized) motion

Beam statistical representation

Statistical representation

Beam: Set of billions (N) of particles evolving with an independent variable s (time, curved abscissa...)

Macro-particle model: $\rightarrow$ Set of n macro-particles ($n < N$)



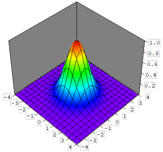
6 coordinates :

- 3 for position: $\mathbf{r}$
  (Cartesian, cylindrical...)
- 3 for motion: $\mathbf{p}$
  (velocity, momentum, energy, slope...)

Distribution function model: $\rightarrow$ function

$$f(\mathbf{r}, \mathbf{p}) \cdot d^3 r \cdot d^3 p$$

Expected number of particles
between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$
between $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$



RMS sizes

Average value of $A(\mathbf{r},\mathbf{r}')$ over the beam:

$$\langle A(\mathbf{r},\mathbf{r}')\rangle = \frac{1}{n}\sum_{i=1}^{n} A(\mathbf{r}_i,\mathbf{r}'_i) = \frac{1}{N}\iint d^3 f(\mathbf{r},\mathbf{r}')\cdot A(\mathbf{r},\mathbf{r}')\, d^3\mathbf{r}'$$

- Examples:

C.o.g position: $(\langle u\rangle, \langle u'\rangle)$

RMS size: $u_{\mathrm{rms}} = \sqrt{\sigma_u} = \sqrt{\left\langle (u-\langle u\rangle)^2\right\rangle}$

RMS slope: $u'_{\mathrm{rms}} = \sqrt{\sigma_{u'}} = \sqrt{\left\langle (u'-\langle u'\rangle)^2\right\rangle}$

RMS coupling: $uu'_{\mathrm{rms}} = \sigma_{uu'} = \langle (u-\langle u\rangle)\cdot(u'-\langle u'\rangle)\rangle$

RMS emittance: $\epsilon_{u,\mathrm{rms}} = \sqrt{u^2_{\mathrm{rms}}\cdot u'^2_{\mathrm{rms}} - (uu'_{\mathrm{rms}})^2}$

Twiss parameters

The ellipse matching the best the beam distribution is:

$$\gamma_{t,u} \cdot u^2 + 2\alpha_{t,u} \cdot u \cdot u' + \beta_{t,u} \cdot u'^2 = A_u$$

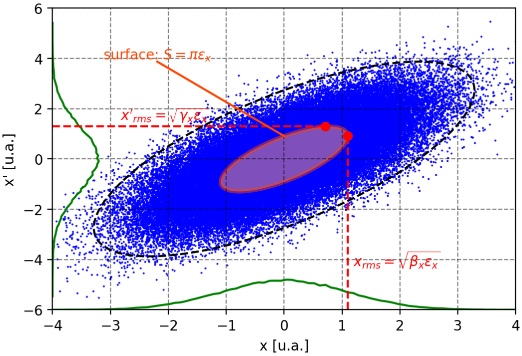
Such as:

$$\beta_{t,u} = \frac{u^2_{\text{rms}}}{\epsilon_{u,\text{rms}}} = \frac{\sigma_u}{\epsilon_{u,\text{rms}}}$$

$$\gamma_{t,u} = \frac{u'^2_{\text{rms}}}{\epsilon_{u,\text{rms}}} = \frac{\sigma_{u'}}{\epsilon_{u,\text{rms}}}$$

$$\alpha_{t,u} = -\frac{uu'_{\text{rms}}}{\epsilon_{u,\text{rms}}} = -\frac{\sigma_{uu'}}{\epsilon_{u,\text{rms}}}$$

Are the beam's Twiss Parameters.



6D model

Particle 6D phase-space coordinates can be:

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix} \qquad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{pmatrix} \text{ or } \begin{pmatrix} x \\ x' \\ y \\ y' \\ \varphi \\ E \end{pmatrix} \qquad \text{for example}$$

Beam distribution can be modelled by a variance-covariance matrix:

$[\sigma]$ such as: $\sigma_{ij} = \left\langle \nu_i \cdot \nu_j \right\rangle$ The sigma matrix.

One has:

$$[\sigma]_e = [T_{e\leftarrow s}] \cdot [\sigma]_s \cdot [T_{e\leftarrow s}]^T$$

$T_{e\leftarrow s}$ is the transfer matrix from s to e.

2. Linear(ized) motion

Envelope equations

Transverse motion equation

Particle dynamics:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{r}, \mathbf{p}; t) \qquad \Rightarrow \qquad \frac{d\mathbf{p}}{ds} = \frac{\mathbf{F}(\mathbf{r}, \mathbf{p}; s)}{\beta_z \cdot c}$$

Magnetic field, no acceleration, transverse motion :

$$\Rightarrow \begin{cases} \dfrac{d\left(x' \cdot \beta_z\right)}{ds} = \dfrac{F_x(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z \cdot m \cdot c^2} \\ \dfrac{d\left(y' \cdot \beta_z\right)}{ds} = \dfrac{F_y(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z \cdot m \cdot c^2} \end{cases}$$

Linac + paraxial approximation: $\beta_x^2 + \beta_y^2 \ll \beta_z^2 \approx \beta^2$

$$\Rightarrow \begin{cases} \dfrac{dx'}{ds} = \dfrac{F_x(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z^2 \cdot m \cdot c^2} = F_x'(\mathbf{r}, \beta, s) \\ \dfrac{dy'}{ds} = \dfrac{F_y(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z^2 \cdot m \cdot c^2} = F_y'(\mathbf{r}, \beta, s) \end{cases}$$

Envelope equation (1)

$$\begin{aligned}
\tilde{x}^2 &= \left\langle x^2 \right\rangle \\
\tilde{x}\tilde{x}' &= \left\langle x x' \right\rangle \\
\tilde{x}'^2 + \tilde{x}\tilde{x}'' &= \left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle \\
\tilde{x}'' &= \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\tilde{x}} - \frac{\tilde{x}'^2}{\tilde{x}} \\
\tilde{x}'' &= \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\tilde{x}} - \frac{\left\langle x x' \right\rangle^2}{\tilde{x}^3} \\
\tilde{x}'' &= \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\left\langle x^2 \right\rangle^{1/2}} - \frac{\left\langle x x' \right\rangle^2}{\left\langle x^2 \right\rangle^{3/2}}
\end{aligned}$$

Envelope equation (2)

RMS size evolution: $$\tilde{x}'' = x''_{\text{rms}} = \frac{\langle x'^2 \rangle + \langle x \cdot x'' \rangle}{\langle x^2 \rangle^{1/2}} - \frac{\langle x \cdot x' \rangle^2}{\langle x^2 \rangle^{3/2}}$$

By noting that: $$\langle x \cdot x'' \rangle = \langle x \cdot F'_x(\mathbf{r}, \beta, s) \rangle$$

One gets: $$\tilde{x}'' - \tilde{K}_x \cdot \tilde{x} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

$$\tilde{\epsilon}_x = \sqrt{\langle x'^2 \rangle \cdot \langle x^2 \rangle - \langle x \cdot x' \rangle^2}$$ The horizontal rms emittance

$$\tilde{K}_x = \frac{\langle x \cdot F'_x(\mathbf{r}, \beta, s) \rangle}{\tilde{x}^2}$$ The force linearisation coefficient

The linearised force can be applied to the envelope equation or as a transfer matrix (with sigma matrix).

2. Linear(ized) motion

Space-charge linearisation

Motion linearisation

$$F_x\left(\mathbf{r},\mathbf{p},s\right)\xrightarrow{\text{linearisation}}k_x\cdot x$$

Interest:

- Easy
- Fast
- Efficient

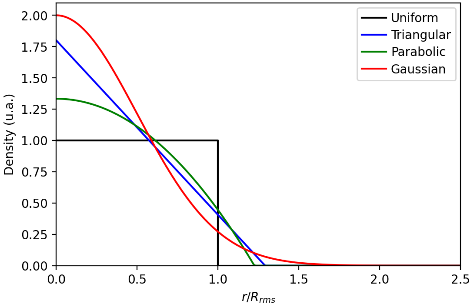
$\Rightarrow$ Equivalent uniform beam

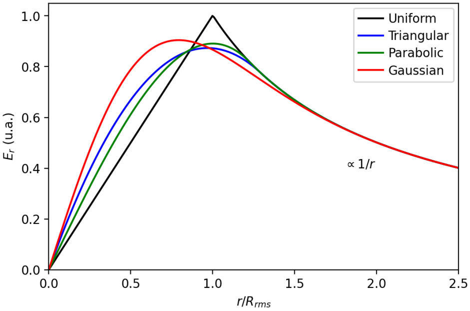
Equivalent beams

Definition of equivalent beams

Two beams are said " equivalent " when they carry the same current (continuous) or charge (bunched) and they have the same sigma matrix.

Example of distribution of continuous equivalent beams:





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RMS emittance evolution

$$\begin{aligned}\frac{d\tilde{\epsilon}_x^2}{ds} &= \frac{d\left\langle x'^2\right\rangle}{ds}\cdot\left\langle x^2\right\rangle + \left\langle x'^2\right\rangle\cdot\frac{d\left\langle x^2\right\rangle}{ds} - 2\left\langle x\cdot x'\right\rangle\cdot\frac{d\left\langle x\cdot x'\right\rangle}{ds} \\ &= 2\left\langle x'\cdot x''\right\rangle\cdot\left\langle x^2\right\rangle + 2\left\langle x'^2\right\rangle\cdot\left\langle x\cdot x'\right\rangle - 2\left\langle x\cdot x'\right\rangle\cdot\left(\left\langle x'^2\right\rangle + \left\langle x\cdot x''\right\rangle\right) \\ &= 2\left(\left\langle x'\cdot x''\right\rangle\cdot\left\langle x^2\right\rangle - \left\langle x\cdot x'\right\rangle\cdot\left\langle x\cdot x''\right\rangle\right)\end{aligned}$$

If the force is linear:

$$x'' = k\cdot x$$

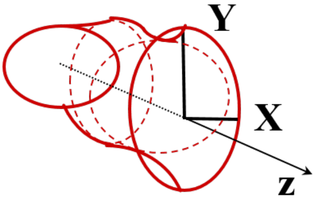
The emittance is constant:

$$\frac{d\tilde{\epsilon}_x^2}{ds} = 2\cdot k\cdot\left(\left\langle x'\cdot x\right\rangle\cdot\left\langle x^2\right\rangle - \left\langle x\cdot x'\right\rangle\cdot\left\langle x^2\right\rangle\right) = 0$$

Rms emittance is conserved in linear force, otherwise it can increase or decrease !

Uniform continuous beam

$$\rho(x,y) = \begin{cases} \rho_0 & \text{if } \frac{x^2}{X^2} + \frac{y^2}{Y^2} < 1 \\ 0 & \text{otherwise} \end{cases}$$



We have: $\begin{cases} \tilde{x} = X/2 \\ \tilde{y} = Y/2 \end{cases}$ and: $\rho_0 = \dfrac{I}{\pi \cdot X \cdot Y \cdot v}$

Space-charge force

$$\begin{cases} \tilde{K}_{SC,x} = \dfrac{q \cdot I}{2\pi\epsilon_0 m \left(\gamma\beta c\right)^3} \cdot \dfrac{2}{X \cdot (X+Y)} = \dfrac{2 \cdot K}{X \cdot (X+Y)} \\ \tilde{K}_{SC,y} = \dfrac{q \cdot I}{2\pi\epsilon_0 m \left(\gamma\beta c\right)^3} \cdot \dfrac{2}{Y \cdot (X+Y)} = \dfrac{2 \cdot K}{Y \cdot (X+Y)} \end{cases}$$

Generalized perveance

$$K = \frac{q \cdot I}{2\pi\epsilon_0 m \left(\gamma\beta c\right)^3}$$

Continuous beam envelope equations

$$\begin{cases} X'' + k_{x,0}^2(s) \cdot X - \dfrac{2K}{X+Y} - \dfrac{\epsilon_{x,\text{eff}}^2}{X^3} = 0 \\ Y'' + k_{y,0}^2(s) \cdot Y - \dfrac{2K}{X+Y} - \dfrac{\epsilon_{y,\text{eff}}^2}{Y^3} = 0 \end{cases}$$

These are the beam 2D envelope equations

$\epsilon_{x,\text{eff}} = 4 \cdot \tilde{\epsilon}_x$ the effective emittance of the continuous beam.

Or, valid whatever the elliptical beam transverse distribution:

Beam RMS 2D envelope equations

$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s) \cdot \tilde{x} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s) \cdot \tilde{y} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \end{cases}$$

A few words about the envelope equation

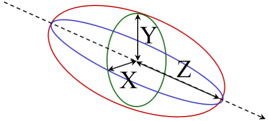
$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s)\cdot\tilde{x} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s)\cdot\tilde{y} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \end{cases}$$

The 2D envelope equation has 3 contributors to the dynamics:

- $k_{x,0}^2(s)\cdot\tilde{x}$: the external force contributor.
- $-\frac{K/2}{\tilde{x}+\tilde{y}}$: the space-charge contributor.
  - The effect is defocusing (negative sign).
  - The effect is proportional to the generalized perveance $K = \frac{q\cdot I}{2\pi\epsilon_0 m(\gamma\beta c)^3}$: it is thus decreasing with energy.
  - The effect decreases with the beam size: the slope of the electric field at the core depends on the beam size.
- $-\frac{\tilde{\epsilon}_x^2}{\tilde{x}^3}$: the emittance contribution. This term increases when the beam size is decreasing: that is even the driver for very small beam sizes (even stronger than space charge).

Bunched uniform beam

$$\rho(x,y) = \begin{cases} \rho_0 & \text{if } \frac{x^2}{X^2} + \frac{y^2}{Y^2} + \frac{z^2}{Z^2} < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \rho_0 = \frac{Q}{\frac{4}{3}\pi \cdot X \cdot Y \cdot Z}$$



We have: $\tilde{u} = U/\sqrt{5}$ $\qquad u = x, y, z$

Space-charge force

$$\begin{cases} F'_{\mathrm{SC},x} = \frac{q}{mc^2} \cdot \frac{3}{4\pi\epsilon_0} \cdot \frac{Q}{\beta^2\gamma^3} \cdot \frac{1-f}{(X+Y)Z} \cdot \frac{x}{X} \\ F'_{\mathrm{SC},y} = \frac{q}{mc^2} \cdot \frac{3}{4\pi\epsilon_0} \cdot \frac{Q}{\beta^2\gamma^3} \cdot \frac{1-f}{(X+Y)Z} \cdot \frac{y}{Y} \\ F'_{\mathrm{SC},z} = \frac{q}{mc^2} \cdot \frac{3}{4\pi\epsilon_0} \cdot \frac{Q}{\beta^2\gamma^3} \cdot \frac{f}{XY} \cdot \frac{z}{Z} \end{cases}$$

$f = f\left(\frac{X}{Y}, \frac{\gamma Z}{\sqrt{XY}}\right)$ is a form factor of the ellipsoid.

3-D space charge parameter

$$K_3 = \frac{q}{5^{3/2} mc^2} \cdot \frac{3}{4\pi\epsilon_0} \cdot \frac{Q}{\beta^2\gamma^3}$$

Bunched beam envelope equations

Beam 3D envelope equations

$$\begin{cases} X'' + k_{x,0}^2(s)\cdot X - \dfrac{K_3(1-f)5^{3/2}}{(X+Y)Z} - \dfrac{\epsilon_{x,\mathrm{eff}}^2}{X^3} = 0 \\ Y'' + k_{y,0}^2(s)\cdot Y - \dfrac{K_3(1-f)5^{3/2}}{(X+Y)Z} - \dfrac{\epsilon_{y,\mathrm{eff}}^2}{Y^3} = 0 \\ Z'' + k_{z,0}^2(s)\cdot Z - \dfrac{K_3 f 5^{3/2}}{XY} - \dfrac{\epsilon_{z,\mathrm{eff}}^2}{Z^3} = 0 \end{cases}$$

$\epsilon_{x,\mathrm{eff}} = 5\cdot\tilde{\epsilon}_x$ the effective emittance of the bunched beam.

Beam RMS 3D envelope equations

$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s)\cdot \tilde{x} - \dfrac{K_3(1-f)}{(\tilde{x}+\tilde{y})\tilde{z}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s)\cdot \tilde{y} - \dfrac{K_3(1-f)}{(\tilde{x}+\tilde{y})\tilde{z}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \\ \tilde{z}'' + k_{z,0}^2(s)\cdot \tilde{z} - \dfrac{K_3 f}{\tilde{x}\tilde{y}} - \dfrac{\tilde{\epsilon}_z^2}{\tilde{z}^3} = 0 \end{cases}$$

Valid whatever the ellipsoidal beam distribution:

2. Linear(ized) motion

Space-charge tune depression

Space-charge tune depression

Replacing the periodic focusing force by a continuous force.
The particle motion without space-charge is:

$$\frac{d^2x}{ds^2} = -\left(\frac{\sigma_{x,0}}{L}\right)^2 \cdot x = -k_{x,0}^2 \cdot x; \qquad k_{x,0} = \left(\frac{\sigma_{x,0}}{L}\right)$$ Phase advance per meter.

$$\Rightarrow x(s) = x_0 \cdot \cos\left(k_{x,0} \cdot s + \varphi\right)$$

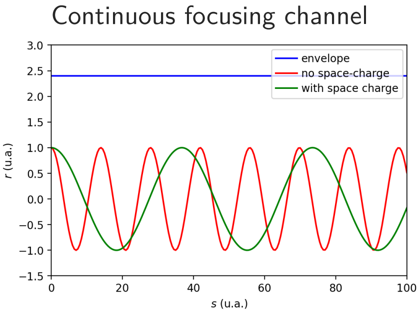
The particle motion with linearised space-charge is:

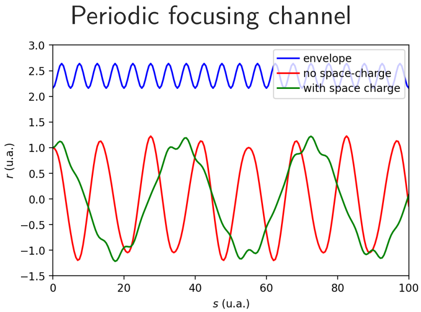
$$\frac{d^2x}{ds^2} = -\left(k_{x,0}^2 - \tilde{K}_{SC,x}\right) \cdot x = -\tilde{k}_x^2 \cdot x$$

$$\tilde{k}_x = \sqrt{k_{x,0}^2 - \tilde{K}_{SC,x}} = \tilde{\eta} \cdot k_{x,0}$$ RMS Phase advance par meter with linear space-charge

$$\eta = \frac{\tilde{k}_x}{k_{x,0}}$$ Space-charge tune depression

Space-charge tune depression (2)





No space-charge:

$$x_{\mathrm{nsc}}(s) = x_0 \cdot \cos\left(k_{x,0} \cdot s + \varphi\right) \qquad x_{\mathrm{nsc}}(s) = \sqrt{\beta_0 \cdot U} \cdot \cos\left(k_{x,0} \cdot s + \varphi\right)$$

Linear space-charge:

$$x_{\mathrm{sc}}(s) = x_0 \cdot \cos\left(\tilde{\eta} \cdot k_{x,0} \cdot s + \varphi\right) \qquad x_{\mathrm{sc}}(s) = \sqrt{\beta_{\mathrm{sc}} \cdot U} \cdot \cos\left(\tilde{\eta} \cdot k_{x,0} \cdot s + \varphi\right)$$

3. Non-linear effects

Tune dispersion

Non-linear motion equation

Motion equation in a linear continuous external force

$$\frac{dx'}{ds} = -k_{x,0}^2 \cdot x + F'_{x,\mathrm{SC}}\left(\mathbf{r}, s\right)$$

The space-charge force can be decomposed:

$$F'_{x,\mathrm{SC}}\left(\mathbf{r}, s\right) = \sum_{i>0} k_{x,\mathrm{SC},i} \cdot x^i$$

We obtain then:

$$\frac{dx'}{ds} = \underbrace{-\left(k_{x,0}^2 - k_{x,\mathrm{SC},1}\right) \cdot x}_{\text{Linear force}} - \underbrace{\sum_{i>1} k_{x,\mathrm{SC},i} \cdot x^i}_{\text{Non-linear part}}$$

Tune dispersion

- At small oscillation amplitude:

$$\frac{dx'}{ds} = -\left(k_{x,0}^2 - k_{x,\mathrm{SC},1}\right) \cdot x = -\left(\eta_{x,c} \cdot k_{x,0}\right)^2 \cdot x$$

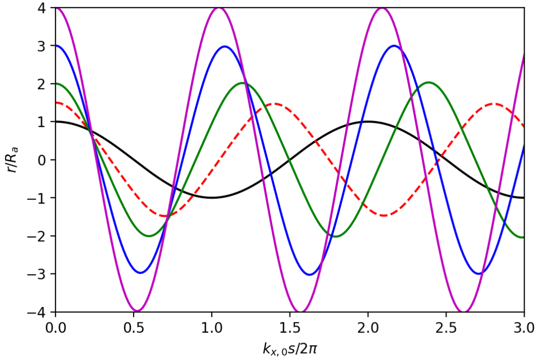
$\eta_{x,c}$: Core space-charge depression.

- At very large amplitudes:
  The particle is often far from the beam it feels essentially the external force. Its oscillation frequency k_x tends to $k_{x,0}$.
- At intermediate amplitude, the particle oscillation frequency depends on its amplitude: this is the space-charge tune dispersion.

$$\boxed{\eta_{x,c} \cdot k_{x,0} \leq k_x < k_{x,0}}$$

Tune depression: example

Particle trajectories around a uniform beam for various amplitudes.



Question

What is the space-charge tune depression here?

3. Non-linear effects

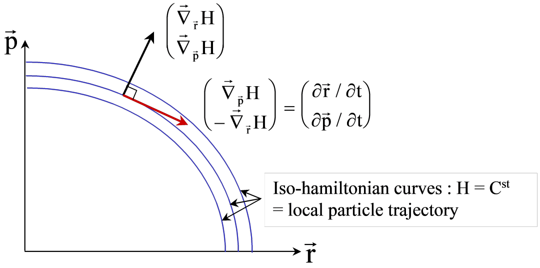
Matching

Motion Hamiltonian

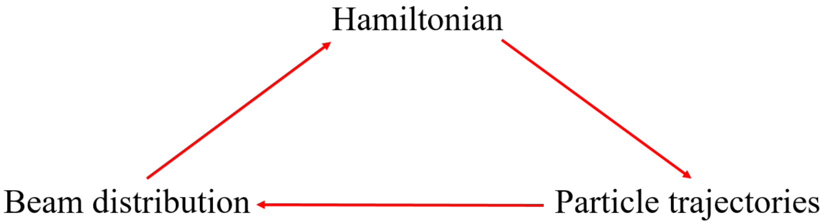
The particle motion can be described with an Hamiltonian H:

$$\begin{cases} \dfrac{d\mathbf{r}}{dt} = \dfrac{\partial H}{\partial \mathbf{p}} = \nabla_{\mathbf{p}} \cdot H \\ \dfrac{d\mathbf{p}}{dt} = -\dfrac{\partial H}{\partial \mathbf{r}} = -\nabla_{\mathbf{r}} \cdot H \end{cases}$$

Particles have phase-space trajectories on which the Hamiltonian is constant (orthogonal to the Hamiltonian gradient)



Perfect matching



Perfect matching: The beam distribution is stationary

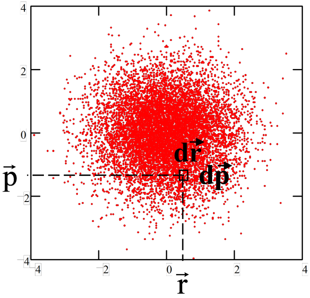
Distribution function

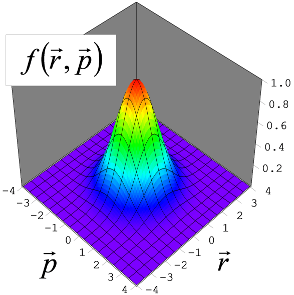
$$f(\mathbf{r}, \mathbf{p}, t) \cdot d\mathbf{r} \cdot d\mathbf{p}$$

is the number of particle at time t in a small phase-space hyper volume $d\mathbf{r} \cdot d\mathbf{p}$ at position $(\mathbf{r}, \mathbf{p})$.

Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f + q\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{p}} f = 0$$





Perfect matching

$$\frac{\partial f}{\partial t} = 0 \qquad \Rightarrow \qquad f\left(\mathbf{r},\mathbf{p}\right)\cdot d\mathbf{r}\cdot d\mathbf{p} = g\left(H\left(\mathbf{r},\mathbf{p}\right)\right)\cdot d\mathbf{r}\cdot d\mathbf{p}$$

But the Hamiltonian depends on the electrostatic potential ϕ and thus on the beam distribution $(\Delta\phi = -\rho/\epsilon_0)$. With:

$$\rho(\mathbf{r}) = \int f\left(\mathbf{r},\mathbf{p}\right) d\mathbf{p}$$

The perfectly matched distribution is then solution of the implicit equation:

$$f\left(\mathbf{r},\mathbf{p}\right) = g\left(H\left(\mathbf{r},\mathbf{p},f\left(\mathbf{r},\mathbf{p}\right)\right)\right)$$

Case study

- Cylindrical continuous beam.
- Radial dynamics only.
- Continuous radial linear focusing channel.

$$\begin{cases} \dfrac{dr}{ds} = r' = \dfrac{\partial H\left(r, r', s\right)}{\partial r'} \\ \dfrac{dr'}{ds} = -k_0^2 \cdot r + F'_{\mathrm{SC}}(r, s) = -\dfrac{\partial H(r, r', s)}{\partial r} \end{cases}$$

$$H(r, r', s) = \frac{1}{2} \cdot r'^2 + \frac{1}{2} \cdot k_0^2 \cdot r^2 + V_{\mathrm{SC}}(r, s)$$

$$V_{\mathrm{SC}}(r, s) \equiv \frac{q\phi(r)}{\beta^2 \gamma^3 mc^2}$$

$$\rho(r) = \int\limits_0^{a'(r)} \int_0^{2\pi} f(H(r, r'))r' dr' d\psi = 2\pi \int\limits_0^{\frac{1}{2}a'(r)^2} f(H(r, r'))d\left(\frac{1}{2}r'^2\right)$$

$$H(r, r') = \frac{1}{2}r'^2 + W(r) \qquad W(r) \equiv \frac{1}{2}k_0^2 \cdot r^2 + V_{\mathrm{SC}}(r, s)$$

$$\frac{1}{2}r'^2 = H(r, r') - W(r) \qquad \frac{1}{2}a'(r)^2 = W(a) - W(r)$$

Case study (2)

$$\rho(r) = 2\pi \int_{W(r)}^{W(a)} f(H)dH$$

$$\forall r < a; \quad \Delta\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi(r)}{dr}\right) = -\frac{2\pi}{\epsilon_0}\int_{W(r)}^{W(a)} f(H)dH$$

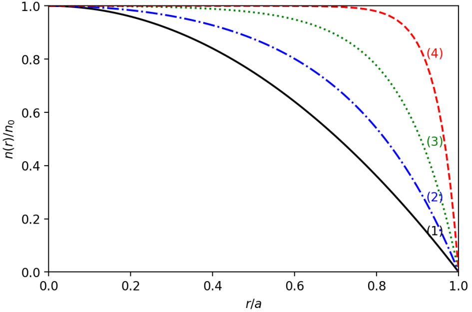
Whatever $f(H)$:

- If emittance dominated ($\eta_c \approx 1$; $V_{SC} \ll k_0^2 r^2$) ("hot" beam),
  - the radial profile depends on f,
  - the particle phase-space trajectories are ellipses.
- If space-charge dominated ($\eta_c \approx 0$; $W(r) \approx 0$ for $r < a$) ("cold" beam),
  - the radial profile tends to uniform,
  - the particle phase-space trajectories tends to rectangular.

Case study – illustration

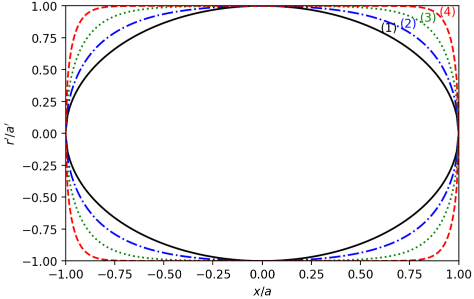
Water-bag beam: $$f(H) = \begin{cases} \frac{1}{H_0} & \text{if } H \leq H_0 \\ 0 & \text{if } H > H_0 \end{cases}$$

Radial density



(1): No space-charge

Phase-space trajectories or distribution contour-plot



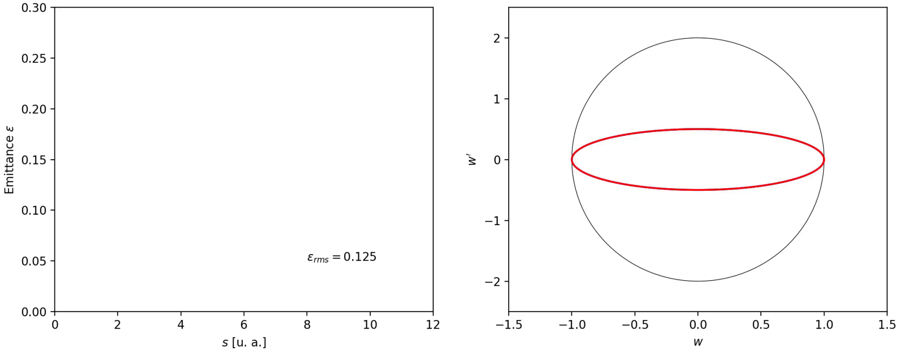
(2)–(4): growing space-charge

3. Non-linear effects

Mismatching

Mismatch - filamentation

The space-charge force is before all non-linear.
A mismatched beam goes filament, and particles are filling gradually the swept phase-space volume.
With associated RMS emittance growth.



Mismatch – 1D mode

Hypothesis: Cylindrical uniform beam.

Envelope equation: $\tilde{x}'' + k_{x,0}^2(s) \cdot \tilde{x} - \dfrac{K}{4\tilde{x}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$

Mismatched beam: $\tilde{x} = \tilde{x}_a\,(1+\delta)$

$$\delta'' + \left(k_{x,0}^2(s) + \frac{K}{4\cdot\tilde{x}_a^2} + 3\frac{\tilde{\epsilon}_x^2}{\tilde{x}_a^4}\right)\cdot\delta = 0$$

$$\delta(s) = M\cdot\cos\left(k_{d,r}s + \varphi\right)$$

Mismatch mode frequency

$$k_{d,r} = \sqrt{k_{x,0}^2(s) + \tfrac{K}{4\cdot\tilde{x}_a^2} + 3\tfrac{\tilde{\epsilon}_x^2}{\tilde{x}_a^4}} = k_{x,0}\sqrt{2\cdot(1+\tilde{\eta}_x^2)}$$

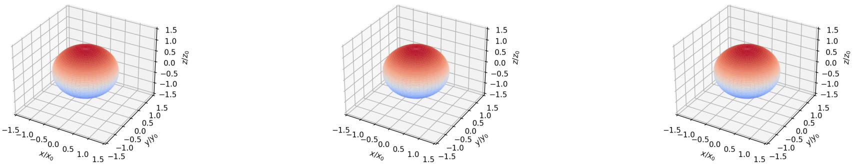
Mismatch – 2D-3D mode

A continuous beam in a quadrupolar channel:

$\Rightarrow$ 2 coupled envelope equations: 2 modes

A bunched beam in a quadrupolar channel and cavities:

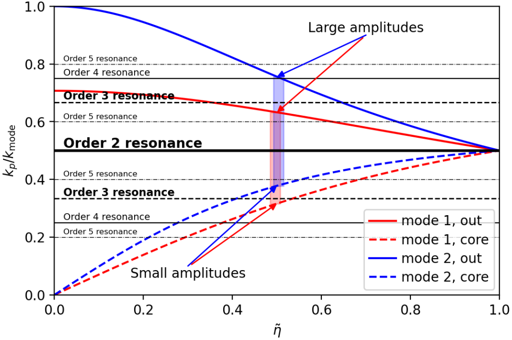
$\Rightarrow$ 3 coupled envelope equations: 3 modes



Second order mismatch-mode parametric resonance

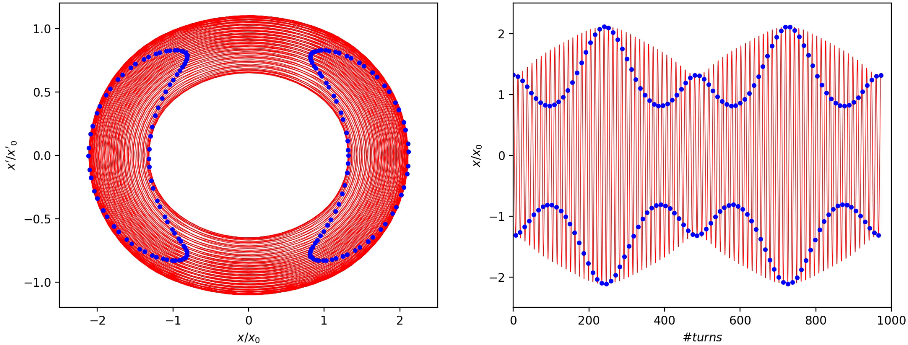
Due to the tune dispersion, there is always a particle amplitude of which the oscillation frequency is half the mismatch mode frequency.

$$\tilde{\eta}_x^2 \cdot k_x^2 < \left(\frac{k_{d,r}}{2}\right)^2 = k_{x,0}^2 \cdot \frac{1+\tilde{\eta}_x^2}{2} < k_{x,0}^2$$



Second order resonance viewing

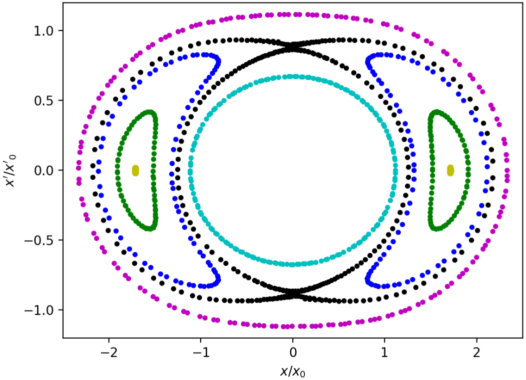
Uniform distribution; Beam mismatch: 10%; $\eta = 0.85$
Particles with an oscillation frequency near half this of a mismatch mode.



Red: particle phase-space trajectory
Blue: stroboscopic viewing at mismatch mode frequency

Second order resonance viewing (2)

Particles with different initial amplitudes (viewing at mismatch frequency):



- No perturbation if large (magenta) or small (cyan) initial amplitudes.
- Stability islands (yellow) for particles at the half mismatch frequency.
- Oscillation around stability island for particles in the black region.

4. Wall effects

Incoherent and incoherent motion

Coherent and incoherent motion

Incoherent motion

The beam consists of many particles, each of which moves inside the beam with its individual betatron amplitude, phase, and even tune Q (under the influence of direct space charge). Amplitude and phase are randomly distributed. The beam and its centre of gravity – and thus the source of the direct space-charge field – do not move (static beam).

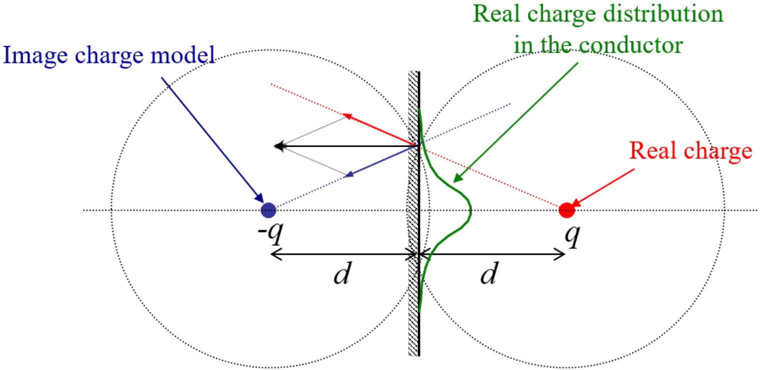
Coherent motion

A static beam is given a transverse fast deflection (< 1 turn) and starts to perform betatron oscillations as a whole. This is readily observed by a position monitor. Note that the source of the direct space charge is now moving: individual particles still continue their incoherent motion around the common coherent trajectory and still experience their incoherent tune shifts as well.

4. Wall effects

Example of an incoherent motion: Plate conductor

Image charge: plate conductor



The real charge q attracts charges in the plate conductor (at a distance d). This charge distribution sets a constant potential in the conductor. It can be modelled by an image charge $-q$ symmetric of the real charge with respect to the plate.

Demonstration of the image charge (plate conductor)

Let us consider a charge q and a perfectly conductor plate at the distance d. We will use a frame centred on the plate (the position of the charge is thus $(d, 0, 0)$. Potential generated by the charge:

$$\phi_q = \frac{q}{4\pi\epsilon_0}\frac{1}{\|\mathbf{r}\|} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left((x-d)^2+y^2+z^2\right)^{1/2}}$$

Let be ϕ_w the potential generated by the wall. The total voltage $\phi_T = \phi_q + \phi_w$ on the electric plate is at the ground voltage $V = 0$. We get:

$$\phi_q(x=0) + \phi_w(x=0) = 0$$

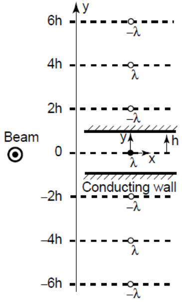
$$\phi_w(x=0) = -\frac{q}{4\pi\epsilon_0}\frac{1}{\left(d^2+y^2+z^2\right)^{1/2}} = \frac{-q}{4\pi\epsilon_0}\frac{1}{\left((0-(-d))^2+y^2+z^2\right)^{1/2}}$$

Image charge

ϕ_w is equivalent to the potential generated by a charge $-q$ at the position $x = -d$.

Electric field if beam between two plates

We consider that the beam pipe is rectangular with a vertical height $2h$ small compared to the width $2w$: $h \ll w$.
We assume a linear distribution. The electric field generated by a linear distribution λ is:



$$\mathbf{E}_\lambda(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0}\frac{x\mathbf{e}_x + y\mathbf{e}_y}{x^2+y^2}$$

The sum of the image charges is then ($y \ll h$):

$$\mathbf{E}(x,y) = \sum_{n=1}^{\infty}\frac{(-1)^n\lambda}{2\pi\epsilon_0}\left[\frac{x\mathbf{e}_x + (2nh-y)\mathbf{e}_y}{x^2+(2nh-y)^2} + \frac{x\mathbf{e}_x - (2nh+y)\mathbf{e}_y}{x^2+(2nh+y)^2}\right]$$
$$\approx \frac{\lambda}{\pi\epsilon_0}\sum_{n=1}^{\infty}(-1)^n\left[\frac{x\mathbf{e}_x - y\mathbf{e}_y}{4n^2h^2} + o(x,y)\right]$$
$$\approx \frac{I}{\beta c\pi\epsilon_0}\frac{\pi^2}{48h^2}\left(-x\mathbf{e}_x + y\mathbf{e}_y + o(x,y)\right)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Tune shift if beam between two plates

Example for a uniform elliptical continuous beam:

$$\begin{cases} \dfrac{dx'}{ds} = -k_{x,0}^2 \cdot x + F'_{x,SC}(\mathbf{r}, s) = -k_{x,0}^2 \cdot x + \dfrac{F_{x,SC}(\mathbf{r}, s)}{\gamma\beta^2 m_0 c^2} \\ \dfrac{dy'}{ds} = -k_{y,0}^2 \cdot y + F'_{y,SC}(\mathbf{r}, s) = -k_{y,0}^2 \cdot y + \dfrac{F_{y,SC}(\mathbf{r}, s)}{\gamma\beta^2 m_0 c^2} \end{cases}$$

$$\begin{cases} \dfrac{dx'}{ds} + \left[k_{x,0}^2 - \dfrac{q}{\pi\epsilon_0 m_0 c^3} \dfrac{I}{\beta^3\gamma} \left(\dfrac{1}{\gamma^2 X \cdot (X+Y)} - \dfrac{\pi^2}{48h^2}\right)\right] \cdot x = 0 \\ \dfrac{dy'}{ds} + \left[k_{y,0}^2 - \dfrac{q}{\pi\epsilon_0 m_0 c^3} \dfrac{I}{\beta^3\gamma} \left(\dfrac{1}{\gamma^2 Y \cdot (X+Y)} + \dfrac{\pi^2}{48h^2}\right)\right] \cdot y = 0 \end{cases}$$

$$\begin{cases} k_{x,inc} = \eta_x k_{x,0} = k_{x,0} \left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3} \dfrac{I}{\beta^3\gamma k_{x,0}^2} \left(\dfrac{1}{\gamma^2 X \cdot (X+Y)} - \dfrac{\pi^2}{48h^2}\right)\right]^{1/2} \\ k_{y,inc} = \eta_y k_{y,0} = k_{y,0} \left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3} \dfrac{I}{\beta^3\gamma k_{y,0}^2} \left(\dfrac{1}{\gamma^2 Y \cdot (X+Y)} + \dfrac{\pi^2}{48h^2}\right)\right]^{1/2} \end{cases}$$

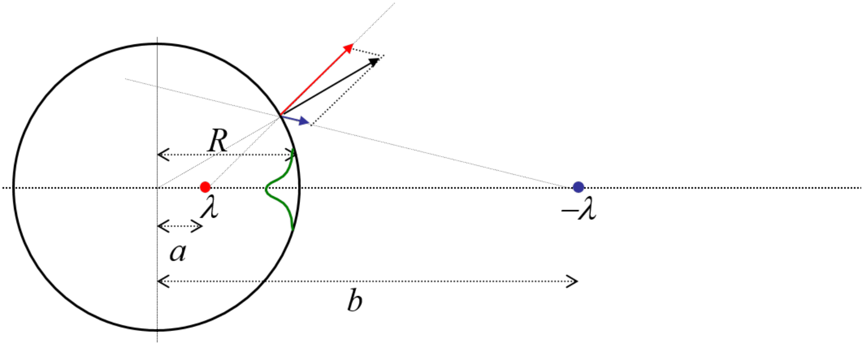
Wall effects against direct space-charge

- The electric image field is vertically defocusing, but horizontally focusing (sign of image term changes), which by the way is not just a feature of this particular geometry, but is typical for most synchrotrons with their rather flattish vacuum pipes;
- The field is larger for small chamber height *h*;
- Image effects decrease with $1/\gamma$, much slower than the direct space-charge term $(1/\gamma^3)$, and thus are of some concern for electron and high-energy proton machines.
- The incoherent motion can be measured by using a quadrupole lens and by introducing a mismatching. The envelope oscillation period gives the incoherent tune by dividing by 2.

4. Wall effects

Example of a coherent motion: Circular conductor

Image charge: cylindrical conductor



The charge distribution on a cylindrical conductor of radius R by a charge per linear meter λ at a distance a from the cylinder center can be modelled by a charge per linear meter $-\lambda$ on the charge-cylinder center axis at distance b such as:

$$a \cdot b = R^2$$

Demonstration of the image charge (cylindrical conductor)

Let us consider a perfectly conductor cylinder of radius R and a linear charge λ at the position $x = a$. The frame center is the center of the cylinder. Electric field generated by the linear charge:

$$\mathbf{E}_q = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz\mathbf{r}}{\|\mathbf{r}\|^3} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r} - a\vec{x}}{x^2 + y^2 + a^2 - 2ax}$$

Let be $\mathbf{E}_w$ the potential generated by the cylinder. The total electric field $\mathbf{E}_T$ on the cylinder is normal to the surface. We get:

$$E_{q,\theta}(r = R) + E_{w,\theta}(r = R) = 0$$

$$E_{w,\theta}(r = R) = -\frac{\lambda}{2\pi\epsilon_0}\frac{a\sin\theta}{R^2 + a^2 - 2aR\cos\theta} = \frac{-\lambda}{2\pi\epsilon_0}\frac{R^2/a\sin\theta}{\left(\left(\frac{R^2}{a}\right)^2 + R^2 - 2\frac{R^2}{a}R\cos\theta\right)}$$

Image charge

The field is equivalent to the one generated by a linear charge $-\lambda$ at the position $x = R^2/a$.

Beam dynamics if offset in a circular pipe

Let us consider a linear distribution with an offset of $\mathbf{r_0} = x_0\mathbf{e}_x + y_0\mathbf{e}_y$. The equivalent charge image of the beam pipe is a linear distribution $-\lambda$ at the position $\mathbf{r_1} = \frac{R^2}{r_0^2}\mathbf{r_0}$. The electric field at the beam center is then:

$$\mathbf{E}_\lambda(\mathbf{r_0}) = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_1}-\mathbf{r_0}}{\|\mathbf{r_1}-\mathbf{r_0}\|^2} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_0}}{R^2-r_0^2} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_0}}{R^2} + o(x_0, y_0)$$

$$\begin{cases} \dfrac{dx_0'}{ds} = -k_{x,0}^2 \cdot x_0 + \dfrac{F_{x,SC}(\mathbf{r}, s)}{\gamma\beta^2 m_0 c^2} = \left[-k_{x,0}^2 + \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma}\dfrac{1}{2R^2}\right]\cdot x_0 \\ \dfrac{dy_0'}{ds} = -k_{y,0}^2 \cdot y_0 + \dfrac{F_{y,SC}(\mathbf{r}, s)}{\gamma\beta^2 m_0 c^2} = \left[-k_{y,0}^2 + \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma}\dfrac{1}{2R^2}\right]\cdot y_0 \end{cases}$$

$$\begin{cases} k_{x,\mathrm{coh}} = \eta_x k_{x,0} = k_{x,0}\left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma k_{x,0}^2}\dfrac{1}{2R^2}\right]^{1/2} \\ k_{y,\mathrm{coh}} = \eta_y k_{y,0} = k_{y,0}\left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma k_{y,0}^2}\dfrac{1}{2R^2}\right]^{1/2} \end{cases}$$

A few features of the coherent tune shift

- The force is linear in $\bar{\mathbf{r}}$, so there is a coherent tune shift.
- The $1/\gamma$ dependence of the tune shift comes from the fact that the charged particles induce the electrostatic field and thus generate a force proportional to their number, but independent of their mass, whereas the deflection of the beam by this force is inversely proportional to their mass $m_0\gamma$.
- The coherent tune shift is never positive.
- Note that a perfectly conducting beam pipe has been assumed here, for simplicity. The effects of a thin vacuum chamber with finite conductivity are more subtle.
- The coherent tune shift can be measured by deflecting the beam with a transverse kicker (with a gate shorter than one revolution period) and by measuring the position (in a ring, turn after turn or in a linac at different positions) with a beam position monitor.

Summary

- Space charge force comes from the charge and current beam distribution: it decreases with energy. At high energy, wall effects (indirect space charge) are greater than direct space charge.
- Space-charge force is non-linear except for uniform distributions.
- Two beams are equivalent if they carry the same current and has the same covariance matrix.
- The envelope equation gives the evolution of the RMS beam size and has 3 contributors: external force, space-charge effect and emittance.
- Space-charge forces increase the motion period: tune depression.
- The non-linearity makes the tune depend on amplitude: tune dispersion.
- To keep the beam distribution, the beam needs a perfect matching.
- If the beam is not matched, beam-size is oscillating.
- Some resonances can occur with stability islands.