TD1, ex 1.3 Covariance of the geodesic eq^n

- $\dot{x}^\mu$ is a 4-vector $\rightarrow$ $\dot{x}'^\mu = (\partial_\alpha x'^\mu)\,\dot{x}^\alpha$ where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$
- diff again $\rightarrow$ $\ddot{x}'^\mu = (\partial_\alpha x'^\mu)\,\ddot{x}^\alpha + (\partial_\beta \partial_\alpha x'^\mu)\,\dot{x}^\alpha \dot{x}^\beta$.
- Remember that

$$\Gamma'^\mu_{\beta\gamma} = \frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\gamma}\Gamma^\varepsilon_{\nu\lambda} + \frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial^2 x^\varepsilon}{\partial x'^\beta \partial x'^\gamma}$$

- So geodesic eq^n $\ddot{x}'^\mu + \Gamma'^\mu_{\beta\gamma}\dot{x}'^\beta \dot{x}'^\gamma$

$$= (\partial_\alpha x'^\mu)\,\ddot{x}^\alpha + (\partial_\alpha \partial_\beta x'^\mu)\,\dot{x}^\alpha \dot{x}^\beta$$

$$+ \left[\frac{\partial x'^\mu}{\partial x^\varepsilon} \underbrace{\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\gamma}}\, \Gamma^\varepsilon_{\nu\lambda} \left(\underbrace{\frac{\partial x'^\beta}{\partial x^\omega}}_{\delta^\nu_\omega} \dot{x}^\omega \underbrace{\frac{\partial x'^\gamma}{\partial x^\delta}}_{\delta^\lambda_\delta} \dot{x}^\delta \right) \right.$$

$$\left. + \frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial^2 x^\varepsilon}{\partial x'^\beta \partial x'^\gamma}\dot{x}'^\beta \dot{x}'^\gamma \right].$$

$$\Rightarrow \quad = (\partial_\alpha x'^\mu)\,\ddot{x}^\alpha + \frac{\partial x'^\mu}{\partial x^\varepsilon}\Gamma^\varepsilon_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda + \text{remainder}$$

where remainder, is the 2nd & 4th terms above, ie

$$\text{remainder} = (\partial_\alpha \partial_\beta x'^\mu)\left(\frac{\partial x^\alpha}{\partial x'^\nu}\right)\left(\frac{\partial x^\beta}{\partial x'^\omega}\right)\dot{x}'^\nu \dot{x}'^\omega \longrightarrow \text{used } \dot{x}^\alpha = \frac{\partial x^\alpha}{\partial x'^\nu}\dot{x}'^\nu$$

$$+ \left[\frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial^2 x^\varepsilon}{\partial x'^\nu \partial x'^\omega}\right]\dot{x}'^\nu \dot{x}'^\omega \longrightarrow \text{renamed } \beta \to \nu,\ \gamma \to \omega$$

Consider $\frac{\partial}{\partial x'^\omega}(\delta^\mu_\nu) = 0 = \frac{\partial}{\partial x'^\omega}\left[\frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial x^\varepsilon}{\partial x'^\nu}\right]$ $\qquad \frac{\partial}{\partial x'^\omega} = \frac{\partial x^\alpha}{\partial x'^\omega}\frac{\partial}{\partial x^\alpha}$

$$= \frac{\partial x'^\mu}{\partial x^\varepsilon}\frac{\partial^2 x^\varepsilon}{\partial x'^\omega \partial x'^\nu} + \frac{\partial x^\varepsilon}{\partial x'^\nu}\overbrace{\frac{\partial x^\alpha}{\partial x'^\omega}\frac{\partial^2 x'^\mu}{\partial x^\varepsilon \partial x^\alpha}}^{\partial'_\omega\left(\frac{\partial x'^\mu}{\partial x^\varepsilon}\right)}$$

$$= 0$$

So these terms give zero $=$ remainder.

$\boxed{\text{TD 1, ex3}}$

$t = \rho\,\mathrm{sh}\psi, \quad x = \rho\,\mathrm{ch}\psi.$

(1) $ds^2 = -dt^2 + dx^2 = -\left(d\rho\,\mathrm{sh}\psi + \rho\cosh\psi\,d\psi\right)^2 + \left(d\rho\cosh\psi + \rho\,\mathrm{sh}\psi\,d\psi\right)^2$

$$= -\left(d\rho^2\,\mathrm{sh}^2 + \cancel{2\,d\rho\,d\psi\,\rho\,\mathrm{sh}\,\mathrm{ch}} + \rho^2\,\mathrm{ch}^2\,d\psi^2\right)$$

$$+ \left(d\rho^2\,\mathrm{ch}^2 + \cancel{2\,d\rho\,d\psi\,\rho\,\mathrm{sh}\,\mathrm{ch}} + \rho^2\,\mathrm{sh}^2\,d\psi^2\right)$$

$$= +d\rho^2 - \rho^2 d\psi^2$$

$$= -\rho^2 d\psi^2 + d\rho^2.$$

→ ψ plays rôle of time as comes with − sign.

(2). $I = -\rho^2\dot\psi^2 + \dot\rho^2$

$S = \int d\tau\left(-\rho^2\dot\psi^2 + \dot\rho^2\right)$

EL eq^n ψ $\Big|\ \frac{d}{d\tau}\left(-\cancel{2}\rho^2\dot\psi\right) = 0$ — (**) $\Rightarrow \rho^2\ddot\psi + 2\rho\dot\rho\dot\psi = 0 \Rightarrow \ddot\psi + \frac{2}{\rho}\dot\rho\dot\psi = 0$

EL eq^n ρ $\Big|\ \frac{d}{d\tau}\left(\cancel{2}\dot\rho\right) = -\cancel{2}\rho\dot\psi^2 \Rightarrow \ddot\rho = -\rho\dot\psi^2$

def^n of proper time $\underbrace{d\tau^2 = -ds^2 = +\rho^2 d\psi^2 - d\rho^2}_{(*)}$ — (A)

(3) from (*) $\rho^2\dot\psi^2 - \dot\rho^2 = 1$

⇒ & from (**) $\rho^2\dot\psi = K$

$\dot\psi = \frac{K}{\rho^2}$

$\Rightarrow \rho^{-2}K^2 - \dot\rho^2 = 1$

$\dot\rho^2 + 1 - K^2/\rho^2 = 0.$

④ $\dot\rho^2 - \frac{K^2}{\rho^2} + 1 = 0.$

$$\dot\rho = \sqrt{\frac{K^2}{\rho^2} - 1} \qquad \dot\rho = \frac{d\rho}{d\tau} = \frac{d\rho}{d\psi}\dot\psi = \rho' \frac{K}{\rho^2}$$

$$\Rightarrow \frac{d\rho}{d\psi} = \frac{\rho^2}{K}\sqrt{\frac{K^2}{\rho^2} - 1} \quad \text{or} \quad \left(\frac{d\rho}{d\psi}\right)^2 = \frac{\rho^4}{K^2}\left(\frac{K^2}{\rho^2} - 1\right)$$

$$= -\frac{\rho^4}{K^2} + \rho^2$$

Set $u = 1/\rho$ then

der

$$\left(\frac{du}{d\psi}\right)^2 = u^2 - \frac{1}{K^2}$$

with sol^n $\boxed{u = 1/\rho = \frac{1}{K}\cosh(\psi - \psi_0)}$

- Why rectilinear motion?

$$K = \rho\cosh(\psi - \psi_0)$$

$$= \rho[\underbrace{\cosh\psi}_{x}\cosh\psi_0 - \sinh\psi\sinh\psi_0]$$

$$K = x\cosh\psi_0 - t\sinh\psi_0$$

$$\Rightarrow x = t\left(\underset{\checkmark}{\tanh\psi_0}\right) + \underset{x_0}{\frac{K}{\cosh\psi_0}}$$

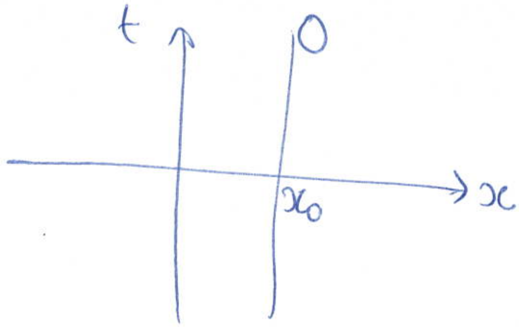
⑤ from Ⓐ

$$d\tau = \sqrt{\rho^2 d\psi^2 - d\rho^2}$$

$$= d\psi\sqrt{\rho^2 - \left(\frac{d\rho}{d\psi}\right)^2}$$

(6) O = Constant x_0

O' = const acc^n.



O: $x = \boxed{\rho \operatorname{ch}\psi = x_0}$

O': why const acc^n this? Want const acc^n in "instantaneous rest frame of O'"

How does one transform to that frame?

Let $u = \frac{dx}{dt}$ = inst. vel of O' at time t.

Then want to transform to a^n inertial frame with velocity $v = u$ at that time t.

What is $\frac{d^2x}{dt^2}$ in that frame?

Step by step. a) $u' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - v\,dx)} = \frac{u-v}{1-uv}$

$\Rightarrow$ instantaneous rest frame indeed $u = v \Rightarrow v' = 0$

b) $a' = \frac{du'}{dt'} = \frac{dt}{dt'}\left(\frac{du'}{dt}\right) = \left[\frac{\frac{du}{dt}}{(1-uv)} + \frac{(u-v)v}{(1-uv)^2}\frac{du}{dt}\right]\frac{dt}{dt'}$

$\frac{du}{dt} = a$, $\frac{dt}{dt'} = \gamma(u)$

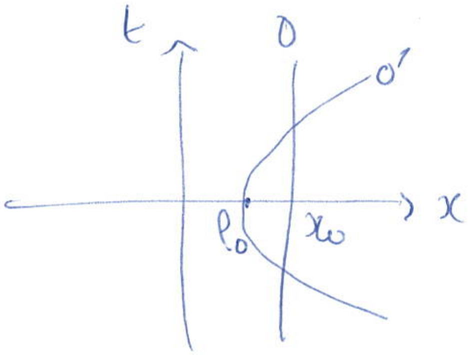
when $u = v$

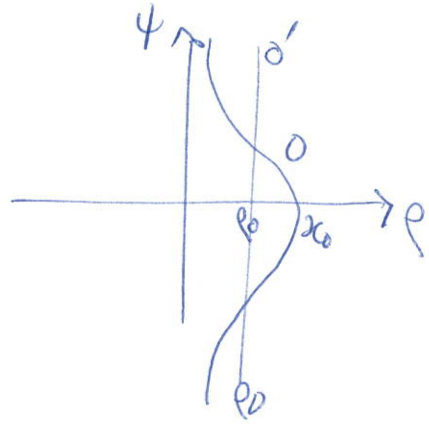
$$a' = \frac{a}{(1-u^2)}\frac{1}{(1-u^2)^{1/2}} = \frac{a}{(1-u^2)^{3/2}}.$$

c) Have a' = const.

$$\Rightarrow a = \frac{a}{(1-u^2)^{3/2}} = \frac{du}{dt}$$

Solve $\Rightarrow x = 1/a' \cosh(a'\tau) \Rightarrow x^2 - t^2 = 1/a'^2$.





$x^2 - t^2 = \rho_0^2$

$\rho^2 = \rho_0^2$

lines of const ρ = curve of acc. obs.

7). For O', $\rho = \text{const} = \rho_0$ Using

$$d\tau = d\psi \sqrt{\rho^2 - \left(\frac{d\rho}{d\psi}\right)^2} \qquad (\text{from } (5))$$

$$\Rightarrow \quad d\tau = d\psi\, \rho_0 \quad \Rightarrow \quad \tau_{0'} = \rho_0 \psi. \qquad (\tau = 0 \text{ at } \psi = 0).$$

For O : $\rho = \frac{x_0}{\text{ch}\psi}$

$$d\tau = d\psi \sqrt{\frac{x_0^2}{\text{ch}^2\psi} - x_0^2 \frac{\text{sh}^2\psi}{\text{ch}^4\psi}}$$

$$\int d\tau = \int \frac{d\psi\; x_0}{\cosh^2\psi}$$

$$\tau_0 = x_0 \tanh\psi$$

8) $x_0 = \rho_0 \cosh\psi_0$ since O has $\rho = \dfrac{x_0}{\cosh\psi}$ and O' has $\rho = \rho_0$, equal when $\psi = \psi_0$

Meet when $\psi = \pm\psi_0$

$$\Delta\tau_0' = 2\rho_0 \psi_0$$

$$\Delta\tau_0 = 2x_0 \tanh\psi_0$$

$$= 2\rho_0 \cancel{\cosh\psi_0} \frac{\sinh\psi_0}{\cancel{\cosh\psi_0}}$$

$$= 2\rho_0 \sinh\psi_0.$$

$$\Rightarrow \frac{\Delta\tau_0}{\Delta\tau_0'} = \frac{\sinh\psi_0}{\psi_0} > 1.$$

9) Light rays: $ds^2 = 0$

$$\Rightarrow \rho^2\dot\psi^2 = \dot\rho^2$$

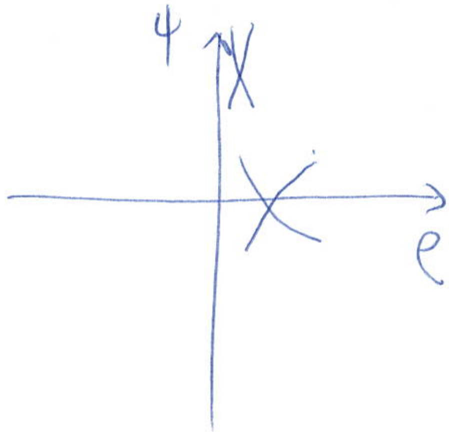
$$\rho\dot\psi = \pm\dot\rho$$

$$\rho = \pm\frac{d\rho}{d\psi}$$

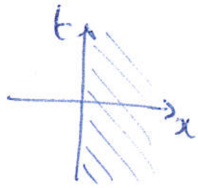
$$\frac{d\rho}{\rho} = \pm d\psi$$

$$\ln\rho/\rho_* = \pm(\psi - \psi_*)$$

$$\rho = \rho_* e^{\pm(\psi-\psi_*)}$$



Td 1, ex 4.



4) $ds^2 = -x^2 dt^2 + dx^2 \qquad -\infty < t < \infty, \quad x > 0.$

1) $ds^2 = 0 \Rightarrow dt = \underset{\text{ingoing}}{\overset{\text{outgoing}}{\pm}} \dfrac{dx}{x} \Rightarrow t = \pm \ln x + c$

$$\boxed{\begin{aligned} u &= t - \ln x \\ v &= t + \ln x \end{aligned}} \quad \begin{matrix} \text{(outgoing)} \\ \text{(ingoing)} \end{matrix} \quad \Rightarrow \begin{cases} t = \dfrac{u+v}{2} \\ \ln x = \dfrac{v-u}{2} \end{cases}$$

$-\infty < u < \infty$
$-\infty < v < \infty.$

$$\Rightarrow \begin{cases} dt = \frac{1}{2}(du + dv) \\ dx = \frac{x}{2}(dv - du) \end{cases}$$

2) $\Rightarrow ds^2 = +\dfrac{x^2}{4}\left[\left(-\cancel{du^2} \mp 2du\,dv \mp \cancel{dv^2}\right) + \left(\cancel{dv^2} - 2du\,dv + \cancel{du^2}\right)\right]$

$= -x^2\, du\, dv$

$$\boxed{ds^2 = -e^{(v-u)}\, du\, dv.}$$

3) $\boxed{\begin{aligned} U &= -e^{-u} \\ V &= e^{v} \end{aligned}}$ $\quad \begin{matrix} U < 0 \\ V > 0. \end{matrix} \Rightarrow \begin{aligned} dU &= e^{-u} du \\ dV &= e^{v} dv \end{aligned} \Rightarrow dU\,dV = e^{(v-u)} du\,dv$

$$\Rightarrow \boxed{ds^2 = -dU\,dV}$$

Then $\begin{aligned} T &= \tfrac{1}{2}(U + V) \\ X &= \tfrac{1}{2}(U - V) \end{aligned} \Rightarrow -dU\,dV = -dT^2 + dX^2 \qquad \begin{aligned} u &= T + X \\ v &= T - X \end{aligned}$

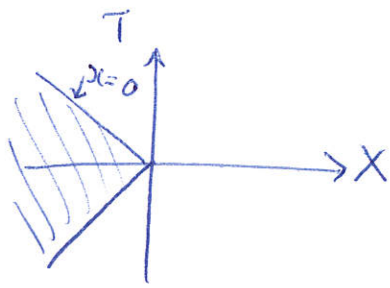
$$\Rightarrow \boxed{ds^2 = -dT^2 + dX^2}$$

Since $U < 0$ & $V > 0 \Rightarrow X < 0$

Also $UV = T^2 - X^2 < 0$

$\Rightarrow T^2 < X^2$

$\Rightarrow |X| < T < |X|$



part of Minkowski Space.

$x = 0 \Rightarrow \begin{matrix} u \to \infty \\ v \to -\infty \end{matrix} \Rightarrow \begin{matrix} U \to 0 \\ V \to 0 \end{matrix} \Rightarrow \begin{matrix} T \to -X \\ T \to +X. \end{matrix}$

But metric extended beyond these ranges has no singularities at all. $x = 0$ is a coord singularity – just a bad choice of coords.

$\boxed{\text{Exam 2018}}$

1) $$ds^2 = +g_{\alpha\beta}\, dx^\alpha dx^\beta = \left(g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\right) d\tilde{x}^\mu d\tilde{x}^\nu$$
$$= \tilde{g}_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu = d\tilde{s}^2$$

2) $$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\nu} V^{\nu}$$
$$\nabla'_\mu V'^\alpha = \partial'_\mu V'^\alpha + \Gamma'^\alpha_{\mu\nu} V'^{\nu} \overset{\text{since it's a tensor}}{=} \frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\left(\partial_\gamma V^\beta + \Gamma^\beta_{\gamma\delta} V^\delta\right)$$

& $\partial'_\mu = \dfrac{\partial x^\gamma}{\partial x'^\mu}\partial_\gamma$; $V'^\nu = \dfrac{\partial x'^\nu}{\partial x^\delta} V^\delta$

$$\Rightarrow \frac{\partial x^\gamma}{\partial x'^\mu}\partial_\gamma\left(\frac{\partial x'^\alpha}{\partial x^\delta}V^\delta\right) + \Gamma'^\alpha_{\mu\nu}\frac{\partial x'^\nu}{\partial x^\delta}V^\delta = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\partial_\gamma V^\beta + \Gamma^\beta_{\gamma\delta}V^\delta\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}$$

since V was arbitrary

$$\text{LHS} = \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}\cancel{V^\delta} + \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x'^\alpha}{\partial x^\delta}\left(\partial_\gamma V^\delta\right) + \Gamma'^\alpha_{\mu\nu}\frac{\partial x'^\nu}{\partial x^\delta}\cancel{V^\delta}$$

$$\text{RHS} = \cancel{\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x'^\alpha}{\partial x^\delta}\left(\partial_\gamma V^\delta\right)} + \Gamma^\beta_{\gamma\delta}\cancel{V^\delta}\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}$$

$$\Rightarrow \Gamma'^\alpha_{\mu\nu}\frac{\partial x'^\nu}{\partial x^\delta} = \Gamma^\beta_{\gamma\delta}\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu} - \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}$$

Now, $\dfrac{\partial x'^\nu}{\partial x^\delta}\dfrac{\partial x^\delta}{\partial x'^\omega} = \delta^\nu_\omega$ so contract each side with $\dfrac{\partial x^\delta}{\partial x'^\omega}$ gives

$$\Gamma'^\alpha_{\omega\mu} = \Gamma^\beta_{\gamma\delta}\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\omega} - \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\omega}\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta} \quad \text{— } (*)$$

1st term is what we want. How about the second one?

From $\dfrac{\partial x'^\alpha}{\partial x^\gamma}\dfrac{\partial x^\gamma}{\partial x'^\mu} = \delta^\alpha_\mu$, diff w r t $\dfrac{\partial}{\partial x^\delta}$

$$\Rightarrow \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}\frac{\partial x^\gamma}{\partial x'^\mu} = -\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x^\delta}$$

Contract each side with $\partial x^\delta / \partial x'^\omega$

$$\Rightarrow \underbrace{\frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\omega}}_{\substack{\uparrow \\ \text{last term on RHS} \\ \text{of } ⊛}} = -\frac{\partial x'^\alpha}{\partial x^\gamma}\left[\frac{\partial x^\delta}{\partial x'^\omega}\frac{\partial}{\partial x^\delta}\left(\frac{\partial x^\gamma}{\partial x'^\mu}\right)\right] = -\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\omega \partial x'^\mu}$$ ②

So ⊛ becomes $\Gamma'^\alpha_{\omega\mu} = \Gamma^\beta_{\gamma\delta}\frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\omega} + \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\omega \partial x'^\mu}$

as required.

③ $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

geodesics from $S = \int d\tau\,(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$ where

$\dot{} = d/d\tau$ with $d\tau^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

EL eq^{n}s: $\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta} \Rightarrow \cancel{2}\ddot\theta = \cancel{2}\sin\theta\cos\theta\,\dot\phi^2$

$\Rightarrow \boxed{\ddot\theta = \sin\theta\cos\theta\,\dot\phi^2}$ — a)

& $\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot\phi}\right) = \frac{\partial L}{\partial\phi} = 0 \Rightarrow \boxed{\sin^2\theta\,\dot\phi = \text{const}}$ — b)

$\downarrow$

or $2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi = 0$

so if $\theta \neq 0$, $2\cos\theta\,\dot\theta\dot\phi + \sin\theta\,\ddot\phi = 0$

- If $\phi = $ const, then $\dot\phi = 0$ so b) satisfied with const $= 0$. & θ undetermined

  but a) says $\ddot\theta = 0 \Rightarrow \theta = c\tau + d$.

  So lines of const ϕ are geodesics along which $\theta = c\tau + d$ with c & d = consts.

- If $\theta = $ const, then from b) $\dot\phi = $ const.

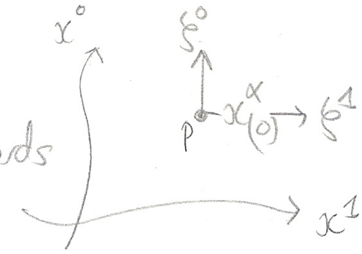
  From a) $\ddot\theta = 0 \Rightarrow \sin\theta\cos\theta = 0 = \frac{1}{2}(\sin 2\theta)$

  $\Rightarrow \sin 2\theta = 0$

  $\Rightarrow 2\theta = 0, \pi$

  $\theta = 0, \pi/2$

$\boxed{\text{Ex 6, TD1}}$



- Want $\Gamma^\mu_{\nu\rho}\big|_0 = 0$, in locally inertial coords ξ^μ.

- Q^n is how does one locally construct the minkowski coords at point p?
  We're given the answer, and asked to check its correct.

$$\xi^\alpha(x) = (x^\alpha - x^\alpha_{(0)}) + \frac{1}{2}(x^\mu - x^\mu_{(0)})(x^\nu - x^\nu_{(0)})\Gamma^\alpha_{(0)\mu\nu} + \dots$$

These are the 1st 2 terms in a taylor expansion.

To simplify things, let's shift the origin of the x^α coords to point p (ie $x^\alpha_{(0)}$)

ie, let's call $y^\alpha = x^\alpha - x^\alpha_{(0)}$, so when $x^\alpha = x^\alpha_{(0)}$, $y^\alpha = 0$

$$\text{So} \quad \xi^\alpha(y) = \underbrace{y^\alpha}_{\uparrow\ \text{zeroth order}} + \underbrace{\frac{1}{2}(y^\mu)(y^\nu)\Gamma^\alpha_{(y=0)\mu\nu}}_{\uparrow\ \text{1st order piece}} + \dots \qquad — ①$$

Can invert this

$$y^\alpha(\xi) = \xi^\alpha - \frac{1}{2}\xi^\mu\xi^\nu\Gamma^\alpha_{(y)\mu\nu} + \dots \qquad — ②$$

↳ I won't bother writing $y=0$, just y

[Can see that easily by dropping indices: ① is $\xi = y + \varepsilon y^2 + \dots$ — ①'
To order $\varepsilon = 0$, $\xi = y$ ie $y = \xi + \varepsilon\xi^{(1)}$
To find $\xi^{(1)}$, subs into ①': $\xi = (\xi + \varepsilon\xi^{(1)}) + \varepsilon(\xi^2) + O(\varepsilon^2)$
So $\xi^{(1)} = -\xi^2$ hence inverse of ①'
is $y = \xi - \varepsilon\xi^2$].

Then, using transf^n of Christoffel, & calculating derivs with ① & ② (primed frame $= \xi^\alpha$, unprimed $= y^\alpha$)

$$\Gamma^\lambda_{(\xi)\mu\nu} = \frac{\partial\xi^\lambda}{\partial y^\rho}\frac{\partial y^\tau}{\partial\xi^\mu}\frac{\partial y^\sigma}{\partial\xi^\nu}\Gamma^\rho_{(y)\tau\sigma} + \frac{\partial\xi^\lambda}{\partial y^\rho}\frac{\partial^2 y^\rho}{\partial\xi^\mu\partial\xi^\nu} \quad \longrightarrow \text{from ②} = -\Gamma^\rho_{(y)\mu\nu}$$

$$= \delta^\lambda_\rho\,\delta^\tau_\mu\,\delta^\sigma_\nu\,\Gamma^\rho_{(y)\tau\sigma} + \delta^\lambda_\rho\left[-\Gamma^\rho_{(y)\mu\nu}\right] + O(y)$$

$$= \cancel{\Gamma^\lambda_{(y)\mu\nu}} - \cancel{\Gamma^\lambda_{(y)\mu\nu}} + O(y) \Rightarrow \boxed{\text{at } y=0,\ \Gamma^\lambda_{(\xi)\mu\nu} = 0}$$

④

$$\left(\delta^\alpha{}_\mu + x^\nu \Gamma^\alpha_{(0)\mu\nu}\right)\left(\delta^\mu{}_\gamma - x^\nu \Gamma^\mu_{(0)\gamma\nu}\right) + \dots$$

$$= \delta^\alpha{}_\gamma - \cancel{x^\nu \Gamma^\alpha_{(0)\gamma\nu}} + \cancel{x^\nu \Gamma^\alpha_{\gamma\nu}} + O(x^2) \qquad = 0$$

But we want $\Gamma'^\alpha_{\beta\gamma}$ at $\xi = 0$ so $x = 0$ in which case (**) is

$$\Gamma'^\alpha{}_{\beta\gamma}\Big|_{\xi=0} = \Gamma^\mu_{(0)\delta\sigma} \delta^\alpha{}_\mu \delta^\rho{}_\beta \delta^\sigma{}_\gamma - \delta^\varepsilon{}_\gamma \delta^\delta{}_\beta \Gamma^\alpha_{(0)\varepsilon\delta}$$

$$= \Gamma^\alpha_{(0)\beta\gamma} - \Gamma^\alpha_{(0)\gamma\beta}$$

$$= 0. \quad \checkmark$$

③ If $\Gamma\big|_{\xi=0} = 0$ then $\dfrac{\partial g_{\mu\nu}}{\partial \xi}\Big|_0 = 0$ since Γ is constructed out of derivs of $g_{\mu\nu}$.

$$\Rightarrow \partial_\alpha\left(g_{\beta\delta}\xi^\beta\xi^\gamma\right) = g_{\beta\delta}\left(\delta^\beta_\alpha \xi^\gamma + \xi^\beta \delta^\gamma_\alpha\right) + \cancel{\left(\partial_\alpha g_{\beta\delta}\right)} \xi^\beta \xi^\gamma$$

$$= 2\, g_{\alpha\beta}\xi^\beta.$$

$\boxed{3}$ 1) $ds^2 = -dt^2 + t^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$ $\quad t>0$

$t' = t\cosh\chi \qquad r' = t\sinh\chi \qquad \Rightarrow \boxed{t'^2 - r'^2 = t^2}$ — ①

$\cancel{g'_{\alpha\beta} = \dfrac{\partial x^\mu}{\partial x'^\alpha}\dfrac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}}$, no complicated.

$$\left.\begin{aligned} dt' &= dt\cosh\chi + r'\,d\chi \\ dr' &= dt\sinh\chi + t'\,d\chi \end{aligned}\right\} \quad \text{want } -dt^2 + t^2 d\chi^2$$

$$dt'^2 = dt^2 \mathrm{ch}^2\chi + 2\,dt\,d\chi\, t \cosh\chi \sinh\chi + r'^2 d\chi^2$$

$$dr'^2 = dt^2 \mathrm{sh}^2\chi + 2\,dt\,d\chi\, t \sinh\chi\cosh\chi + t'^2 d\chi^2$$

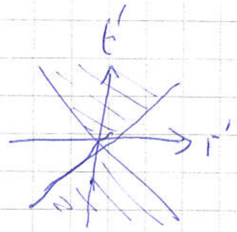
$\Rightarrow dr'^2 - dt'^2 = -dt^2 + t^2 d\chi^2$ where have used ①

So $ds'^2 = -dt'^2 + dr'^2 + r'^2(d\theta'^2 + \sin^2\theta\, d\phi'^2)$.

This is part of Minkowski space. Since $t' > 0$

$$\Rightarrow t'^2 - r'^2 > 0$$

$$\Rightarrow t'^2 > r'^2$$



In Minkowski geodesics are straight lines:

$$r' = r_0' + v t' \qquad r_0' \,\&\, v = \text{consts}$$

$$\Rightarrow t \sinh X = r_0' + v\, t \cosh X$$

$$\Rightarrow t(\sinh X - v\cosh X) = r_0'$$

$$\Rightarrow t = \frac{r_0'}{\sinh X - v\cosh X}.$$