GENERAL RELATIVTY NPAC

TD 1

1 General coordinate transformations in Minkowski space

1. Start from Minkowski coordinates $\xi^{\mu} = (t, x, y, z)$ with metric $\eta_{\mu\nu}$. On transforming to general curvilinear coordinates x^{μ} , the metric tensor and Christoffel symbols are defined by

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}$$
 (1)

$$\Gamma^{\mu}{}_{\nu\lambda}(x) = \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial^{2} \xi^{\alpha}}{\partial x^{\nu} \partial x^{\lambda}}$$
 (2)

Show that

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} (\partial_{\nu} g_{\kappa\lambda} + \partial_{\lambda} g_{\kappa\nu} - \partial_{\kappa} g_{\nu\lambda}). \tag{3}$$

What are the symmetries of $\Gamma^{\mu}_{\nu\lambda}$?

2. Show that under a coordinate transformation $x^{\mu} \to x'^{\mu}$ (assumed invertible),

$$\frac{d^2x'^{\alpha}}{d\tau^2} + \Gamma'^{\alpha}{}_{\beta\gamma}\frac{dx'^{\beta}}{d\tau}\frac{dx'^{\gamma}}{d\tau} = \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\right)\left[\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}\right]. \tag{4}$$

Hence show that if the geodesic equation holds in one set of coordinates, it holds in another.

3. Determine how the Christoffel symbols transform under a coordinate transformation $x^{\mu} \to x'^{\mu}$.

2 Geodesic equation

1. Consider a time-like curve $C(\lambda)$, parametrised by a parameter λ , on a space-time with metric $g_{\mu\nu}$. What is the sign of $g_{\alpha\beta}(x)\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}$ on this curve? Obtain the geodesic equation by minimising the proper-time between two points $p_0 = C(\lambda_0)$ and $p_1 = C(\lambda_1)$:

$$S_0[x] = -m \int_{p_0}^{p_1} d\tau = -m \int_{p_0}^{p_1} \frac{d\tau}{d\lambda} d\lambda = -m \int_{p_0}^{p_1} d\lambda \sqrt{-g_{\alpha\beta}(x) \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}}.$$
 (5)

(Here τ is the proper-time.) In the last step choose $\lambda = \tau$ to express the geodesic equation in terms of x^{μ} and $\dot{x}^{\mu} = dx^{\mu}/d\tau$.

2. Show that the same geodesic equation is obtained from the action

$$S_1[x] = \int d\tau \mathcal{L}[x^{\mu}, \dot{x}^{\mu}] \tag{6}$$

where

$$\mathcal{L} = g_{\mu\nu}(x) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \tag{7}$$

Note: For massless particles, proper-time does not exist. The geodesic equation is expressed in terms of a parameter λ along the light-like geodesic, satisfying

$$g_{\alpha\beta}(x)\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0.$$
 (8)

The geodesic equation then reads

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\lambda}}{d\lambda} = 0 \tag{9}$$

3. Consider a space-time metric of the form

$$ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + r^{2}d\Omega^{2}$$
(10)

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. This is the general form of a static spherically symmetric metric, which we will use later in the course to describe the gravitational field of a star. Write down the corresponding Lagrangian \mathcal{L} . Show that t is a cyclic variable, and determine its equation of motion. From this, show that

$$\Gamma_{rt}^t = \Gamma_{tr}^t = \frac{A'}{2A}, \qquad \Gamma_{\mu\nu}^t = 0 \text{ otherwise.}$$
(11)

From the (r, θ, ϕ) equation determine the remaining Christoffel symbols. Verify you calculations by determining the Christoffel symbols directly from the metric.

3 Rindler coordinates

Rindler coordinates (ρ, ψ) are defined in terms of Minkowski coordinates (t, x) by

$$t = \rho \sinh \psi \qquad x = \rho \cosh \psi \tag{12}$$

- 1. Write down the metric in Rindler coordinates, and determine all the non-vanishing Christoffel symbols.
- 2. Write down the ρ and ψ components of the geodesic equation, together with the definition of proper-time expressed in Rindler coordinates.
- 3. Show that a first integral of the ψ -geodesic equation is $\rho^2 \dot{\psi} = K$ where K is a positive integration constant. Now show that ρ satisfies

$$\dot{\rho}^2 - \frac{K^2}{\rho^2} + 1 = 0 \tag{13}$$

4. The trajectories of the geodesics in space-time are of the form $\rho(\psi)$. Eliminate τ to find an equation for $d\rho/d\psi$. Verify that its solution is

$$\rho^{-1} = \frac{1}{K} \cosh(\psi - \psi_0) \tag{14}$$

where ψ_0 is an integration constant. Show that this corresponds to rectilinear motion of the form $x = x_0 + vt$ where (t, x) are Minkowski coordinates.

5. Show that the proper-time for an observer is given by

$$\tau = \int d\psi \sqrt{\rho^2 - \left(\frac{d\rho}{d\psi}\right)^2} \tag{15}$$

6. Consider two observers: \mathcal{O} who is intertial and fixed at $x = x_0 > 0$; and \mathcal{O}' who has constant acceleration (that is, in her instantaneous rest frame, the acceleration is constant). Show that the trajectory of \mathcal{O}' is given by

$$\rho = \rho_0 \,, \qquad x^2 - t^2 = \rho_0^2 \tag{16}$$

and determine her acceleration in terms of ρ_0 . Draw the world-lines of \mathcal{O} and \mathcal{O}' on the space-time diagrams (t,x) and then (ψ,ρ) . Indicate x_0 and ρ_0 on each of your diagrams.

- 7. Use (15) to deterime the proper-time of \mathcal{O} and \mathcal{O}' as a function of ψ .
- 8. Now introduce a constant ψ_0 such that $x_0 \equiv \rho_0 \cosh \psi_0$. Calculate the proper-time which has elapsed between the two instances at which the observers meet (namely $\psi = \pm \psi_0$). Show that

$$\frac{\Delta \tau_{\mathcal{O}}}{\Delta \tau_{\mathcal{O}'}} = \frac{\sinh \psi_0}{\psi_0} > 1 \tag{17}$$

Which is larger?

9. Show that the trajectories $\rho(\psi)$ of light rays are given by

$$\rho = \rho_* e^{\pm(\psi - \psi_*)} \tag{18}$$

where ψ_* and ρ_* are the coordinates of a point on the light-ray.

4 Extension of the Rindler metric

Here we consider the metric

$$ds^{2} = -x^{2}dt^{2} + dx^{2}, \qquad -\infty < t < +\infty, \qquad x > 0.$$
(19)

Notice that, in this coordinate system, the metric is singular at x = 0.

1. Write down the equation giving trajectories t(x) of light-rays in this metric. Express them in terms of the new coordinates

$$u \equiv t - \ln(x)$$
 $v \equiv t + \ln(x)$

- 2. Write the metric in this new coordinate system (u, v).
- 3. Now carry out a further change of variables

$$U = -e^{-u}, \qquad V = e^v$$

Write down the metric in these variables. Same question for the change of variables

$$T = \frac{1}{2}(U+V), \qquad X = \frac{1}{2}(U-V)$$

Identify this new metric. In what range are the coordinates T and X defined? What can you say about the singularity at x=0 in the metric (19)? Convince yourself that it is just a "coordinate singularity", namely due to an inadapted choice of coordinates, and that the metric written in another set of coordinates is perfectly well defined.

5 From past exam: Basics

1. Show that the spacetime interval $ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ is invariant under coordinate transformations $x^{\alpha} \to \tilde{x}^{\alpha}$ if $g_{\alpha\beta}$ are components of a tensor transforming according to the tensor transformation law

$$g_{\alpha\beta} \longrightarrow \tilde{g}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}} g_{\mu\nu}.$$

2. Let V^{α} be the contravariant components of a vector, and consider an invertible coordinate transformation $x^{\delta} \to \tilde{x}^{\delta}$. Write down the transformation law for $\nabla_{\beta}V^{\alpha}$, and deduce that Christoffel symbols transform according to

$$\Gamma^{\alpha}_{\beta\gamma} \longrightarrow \tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\mu}_{\rho\sigma} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\beta}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\gamma}} + \frac{\partial^{2} x^{\sigma}}{\partial \tilde{x}^{\beta} \tilde{x}^{\gamma}} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\sigma}}.$$

3. Consider a 2-sphere with coordinates (θ, ϕ) and line-element

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Show that lines of constant longitude (ϕ =constant) are geodesics, and that the only line of constant latitude (θ =constant) that is a geodesic is the equator ($\theta = \pi/2$).

6 From Exam: Locally inertial coordinates

- 1. At a point $x_{(0)}^{\alpha}$ in some coordinate system x^{α} , and as seen in lectures, it is always possible to construct a *locally inertial coordinate system* ξ^{α} . Which quantity should vanish at $x_{(0)}^{\alpha}$ in this locally inertial coordinate system, and why?
- 2. Suppose that the point $x^{\alpha}_{(0)}$ and in the coordinate system x^{α} , the Christoffel symbol has the value $\Gamma^{\alpha}_{(0)\mu\nu}$. Then at $x^{\alpha}_{(0)}$, the ξ^{α} are constructed as follows:

$$\xi^{\alpha}(x) = x^{\alpha} - x_{(0)}^{\alpha} + \frac{1}{2} \left(x^{\mu} - x_{(0)}^{\mu} \right) \left(x^{\nu} - x_{(0)}^{\nu} \right) \Gamma_{(0)\mu\nu}^{\alpha}. \tag{20}$$

The point $x_{(0)}^{\alpha}$ in the new coordinates is the origin $\xi^{\alpha} = 0$. Prove explicitly that, when transformed to the new coordinates, the relevant quantity that should vanish at $\xi^{\alpha} = 0$ indeed does so.

[Hint: for simplicity, choose the origin of your x^{α} coordinates such that $x_{(0)}^{\alpha} = 0$.]

3. In the locally inertial coordinate system ξ^{α} , show that $\partial_{\alpha}(g_{\beta\gamma}\xi^{\beta}\xi^{\gamma}) = 2g_{\alpha\beta}\xi^{\beta}$.

7 From past exam : Coordinate transformations

Consider the line element

$$ds^{2} = -dt^{2} + t^{2}[d\chi^{2} + \sinh^{2}\chi(d\theta^{2} + \sin^{2}\theta d\phi^{2})], \qquad t > 0.$$

Carry out the change of coordinates

$$\tilde{t} = t \cosh \chi, \qquad \tilde{r} = t \sinh \chi, \qquad \tilde{\theta} = \theta, \qquad \tilde{\phi} = \phi.$$
 (21)

Identify the new metric, specifying carefully the allowed ranges of the coordinates \tilde{t} and \tilde{r} . What do geodesics look like in this new metric (note: essentially no calculation is required to answer this question)? Conclude that in the original (t, χ, θ, ϕ) coordinate system, geodesics are given by $t = d/(\sinh \chi - v \cosh \chi)$ where v is a constant that can be interpreted as a speed, and d is another constant that can be interpreted as an initial position.