Université Paris Cité 2023-2024
D.A. Steer

GENERAL RELATIVTY
NPAC

TD 2

1 Useful identities to prove

Prove the following useful relations :

$$\begin{aligned} \nabla_\gamma g_{\alpha\beta} &= 0 \, , \\ g_{\alpha\mu}\partial_\gamma g^{\mu\beta} &= -g^{\mu\beta}\partial_\gamma g_{\alpha\mu} \, , \\ \partial_\gamma g^{\alpha\beta} &= -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha} \, . \end{aligned}$$

Other very useful relations are given (and proved) in the boxed equations in the next exercise.

2 Tensor densities

Recall that under a change of coordinates $x^\mu \to x'^\mu(x^\alpha)$, scalars are invariant : that is for a scalar A, the transformation law is $A \to A' = A$. Vectors transform as

$$V^\alpha \to V'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta$$

and tensors as

$$T^{\alpha\gamma} \to T'^{\alpha\gamma} = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x'^\gamma}{\partial x^\delta} T^{\beta\delta}$$

• *Scalar densities*, on the other hand, are **defined** to transform as

$$\boxed{\mathcal{A} \to \mathcal{A}' = \left|\frac{\partial x}{\partial x'}\right| \mathcal{A}} \tag{1}$$

where

$$\left|\frac{\partial x}{\partial x'}\right| = \det(J^\alpha{}_\beta) \qquad \text{where} \qquad J^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial x'^\beta} \, .$$

That is, $J_{\alpha\beta}$ is the Jacobian (matrix) associated with the coordinate transformation.
• *Vector densities* are **defined** to transform as

$$\boxed{\mathcal{V}^\alpha \to \mathcal{V}'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} \mathcal{V}^\beta \left|\frac{\partial x}{\partial x'}\right|},$$

and similarly for tensor densities. The aim of this exercise is to get some familiarity with these densities, as the action for GR is written in terms of these (and you will understand why below).

1. Let $M_{\alpha\beta}$ any 2nd-rank covariant tensor. From its transformation law, deduce that

$$(\det M_{\alpha\beta})^{1/2}$$

is a scalar density.

2. Taking $M_{\alpha\beta}$ to be the metric, and on denoting

$$\boxed{g \equiv \det g_{\alpha\beta}}$$

with $g < 0$, deduce that $\sqrt{-g}$ is a scalar density. Hence conclude that if A is a scalar, then $\mathcal{A} = \sqrt{-g}A$ is a scalar density.

3. When seeking an action $S = \int d^4x\mathcal{L}$ for Einstein's equations, S must be a scalar (why ?). Deduce, on using the definition (1), that $\mathcal{L}$ must be a scalar density (known as the *Lagrangian density*).

   Thus we can write $\mathcal{L} = \sqrt{-g}\Phi$ where Φ is a scalar, so that $S = \int d^4x\sqrt{-g}\Phi$. Show that $d^4x\sqrt{-g(x)}$ is an invariant measure (known as the *space-time volume element*).

4. **Definitions** : Let A, C^μ and B_μ be respectively a scalar, a contravariant vector and a covariant vector. Then the corresponding densities are defined by

$$\begin{aligned} \mathcal{A} &= \sqrt{-g}A && \text{scalar density} \\ \mathcal{C}^\mu &= \sqrt{-g}C^\mu && \text{contravariant vector density} \\ \mathcal{B}_\mu &= \sqrt{-g}B_\mu && \text{covariant vector density} \end{aligned}$$

The aim of the next questions is to learn how to take covariant derivatives of different (scalar/vector/tensor) densities.

5. Derivatives of $\sqrt{-g}$. Using the identity $\det M = e^{\mathrm{tr}\ln M}$ where M is a matrix, show that

$$\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta} \tag{2}$$

Hence deduce that

$$\boxed{\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}} \tag{3}$$

6. Show, using (3) that

$$\boxed{\partial_\mu(\sqrt{-g}) = \sqrt{-g}\Gamma^\alpha_{\mu\alpha}}$$

and that

$$\boxed{\sqrt{-g}g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\partial_\beta(\sqrt{-g}g^{\alpha\beta})}$$

Both these are *extremely* useful relations.

7. Show that

$$\nabla_\alpha\mathcal{A} = \partial_\alpha\mathcal{A} - \Gamma^\beta_{\alpha\beta}\mathcal{A}$$

8. Now consider a vector density $\mathcal{V}^\alpha$. Show that

$$\nabla_\alpha\mathcal{V}^\beta = \partial_\alpha\mathcal{V}^\beta + \Gamma^\beta_{\alpha\gamma}\mathcal{V}^\gamma - \Gamma^\gamma_{\alpha\gamma}\mathcal{V}^\beta$$

9. Hence show that

$$\boxed{\nabla_\alpha\mathcal{V}^\alpha = \partial_\alpha\mathcal{V}^\alpha} \tag{4}$$

Deduce that for a vector

$$\boxed{\nabla_\alpha V^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}V^\alpha)}$$

This is again an extremely useful identity.

3 Parallel transport

On a curve $x^\alpha(\lambda)$ with tangent vectors $t^\alpha = dx^\alpha/d\lambda$, a vector v^μ is said to be 'parallel transported' if it satisfies

$$t^\alpha \nabla_\alpha v^\mu = 0 \tag{5}$$

More generally, parallel transport of a tensor $T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_p}$ is defined by

$$t^\alpha \nabla_\alpha T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_p} = 0$$

1. Show that (5) is equivalent to

$$\frac{Dv^\sigma}{d\lambda} \equiv \frac{dv^\sigma}{d\lambda} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\lambda} v^\nu = 0 \tag{6}$$

   Is it true to say that a geodesic is a curve along which its tangent vector is parallel transported ?

2. What does (6) reduce to in Minkowski space ? Comment.

   In the following we consider 2 different metrics in 2D-space :

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 && \text{2D euclidean plane in polar coordinates} \\ ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 && \text{2D surface of a sphere} \end{aligned}$$

3. In the euclidean plane, consider a circle of radius 5 centered on the origin and described by the parameter $\lambda = \theta$. Now consider the parallel transport of a vector around this circle, starting at the point $(r, \theta) = (5, 0)$ where the vector is taken to have components $(v^r, v^\theta) = (1, 0)$; and finishing at the point $(r, \theta) = (5, 2\pi)$. By an explicit calculation show that the vector (v^r, v^θ) at the final point is unchanged.

4. On the surface of the sphere, consider a circle parametrised by

$$\theta = \theta_0, \qquad \phi = \phi_0 + \lambda$$

   Write down the tangent to the curve, and show that equation (6) takes the form

$$\frac{dv^\theta}{d\lambda} - v^\phi \sin\theta_0 \cos\theta_0 = 0, \qquad \frac{dv^\phi}{d\lambda} + v^\theta \mathrm{cotan}\theta_0 = 0$$

   Combine these into one second order equation for v^θ, which you can then integrate. Taking as initial conditions $(v^\theta, v^\phi) = (1, 0)$ show that the solution is

$$v^\theta(\lambda) = \cos[\lambda \cos(\theta_0)] , \qquad v^\phi(\lambda) = -\sin[\lambda\cos(\theta_0)]/\sin\theta_0$$

   When the final point ϕ_1 is given by $\phi_1 = \phi_0 + 2\pi$, the initial and final points coincide. Deduce $(v^\theta(2\pi), v^\phi(2\pi))$ and show that this is not equal to (1,0) – unless one is on the equator, $\theta = \pi/2$.

4 Geodesic deviation equation : a covariant derivation

Consider a continuous sequence of time-like geodesics parametrised by propertime τ. Each geodesic is labelled by a parameter μ. This is sometimes called a *congruence* of timelike geodesics, and the entire congruence can be described by the parametric equations

$$x^\alpha = r^\alpha(\tau, \mu).$$

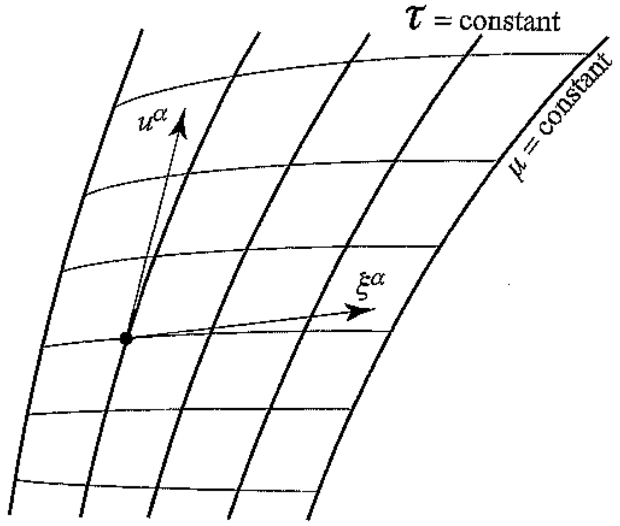

FIGURE 1 – Congruence of timelike geodesics

When μ is fixed and τ varies one goes along a selected geodesic in the congruence, and the geodesics tangent vector is

$$u^\alpha = \partial r^\alpha / \partial \tau.$$

When τ is fixed and μ varied, the displacement is across geodesics.

The vector

$$\xi^\alpha := \partial r^\alpha / \partial \mu$$

is called that *deviation vector* that points from geodesic to geodesic, see figure. The aim is to derive an evolution equation for this deviation vector.

1. Convince yourself that the geodesic equation can be expressed as $u^\beta \nabla_\beta u^\alpha = 0$.
2. Show that the definitions of u^α and ξ^α imply that

$$\xi^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta \xi^\alpha = 0$$

   and that the equation can be re-expressed in the covariant form

$$\xi^\beta \nabla_\beta u^\alpha - u^\beta \nabla_\beta \xi^\alpha = 0.$$

3. Using the definition of the Riemann tensor in terms of the commution of 2 covariant derivatives, show that one can write

$$\xi^\gamma u^\delta (\nabla_\gamma \nabla_\delta u^\alpha - \nabla_\delta \nabla_\gamma u^\alpha) = R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta$$

4. Now rewrite the first 2 terms on the LHS (using the geodesic equation). You should arrive at an expression of the form

$$-R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta = u^\delta \nabla_\delta (u^\gamma \nabla_\gamma \xi^\alpha) - [u^\delta \nabla_\delta \xi^\gamma - \xi^\delta \nabla_\delta u^\gamma](\nabla_\gamma u^\alpha)$$

   Convince yourself that the 2nd term, the one in square brackets, vanishes.
5. Finally show that you can rewrite this equation in the form

$$\boxed{\frac{D^2 \xi^\alpha}{D\tau^2} = -R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta} \qquad (7)$$

   which is the equation of geodesic deviation. Notice that there is a relative acceleration between geodesics whenever the space-time is curved, that is whenever the Riemann curvature is non-zero.

5 Bianchi Identity

We wish to prove the Bianchi identity of the Riemann curvature tensor :

$$\nabla_{[a}R_{bc]d}{}^{e} = 0 \tag{8}$$

where for any tensor T_{ijk}, the totally antisymmetric tensor $T_{[ijk]}$ is defined by

$$T_{[ijk]} = \frac{1}{3!}(T_{ijk} - T_{jik} + T_{jki} - T_{kji} + T_{kij} - T_{ikj}) \tag{9}$$

Let V_b be a general co-vector.

1. Recall how $(\nabla_b\nabla_c - \nabla_c\nabla_b)V_d$ is expressed in terms of the Riemann tensor. Deduce $\nabla_a(\nabla_b\nabla_c - \nabla_c\nabla_b)V_d$ in terms of the Riemann tensor, the co-vector, and their covariant derivatives.
2. Explain why
$$(\nabla_a\nabla_b - \nabla_b\nabla_a)(\nabla_c V_d) = R_{abc}{}^{e}\nabla_e V_d + R_{abd}{}^{f}\nabla_c V_f \tag{10}$$
Hint : use the fact that $\nabla_c V_d$ is a tensor.
3. After antisymmetrizing over a, b and c the equations obtained in 1. and 2. , infer that
$$R_{[abc]}{}^{e}\nabla_e V_d + R_{[ab|d|}{}^{f}\nabla_{c]}V_f = V_e\nabla_{[a}R_{bc]d}{}^{e} + R_{[bc|d|}{}^{e}\nabla_{a]}V_e \tag{11}$$
where the vertical bars indicate that we do not anti-symmetrize over d. Deduce that
$$V_e\nabla_{[a}R_{bc]d}{}^{e} = 0 \tag{12}$$
from which we arrive at (8) since V_e is a general co-vector.

6 Einstein Hilbert action

In the absence of matter, $T^{\mu\nu} = 0$, the Einstein equations are $G_{\mu\nu} = 0$. As discussed in exercise 2, an action yielding this equation should be of the form

$$S \propto \int d^4x\sqrt{-g} \times (\text{scalar depending on}\, g_{\mu\nu}) \tag{13}$$

Possible scalars are constants, R (Ricci scalar) ; R^2 ; $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ etc. However, Einstein's equations are second order in derivatives of g. It is a somewhat subtle point that requires some thought — beyond the scope of this particular exercise (though you are welcome to think about it !, and if there is time we may mention it in the context of modified gravity) — but this means that the scalar in question can only contain R. In fact, the appropriate choice is summed up in the *Einstein-Hilbert action*

$$\boxed{S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R}$$

From now on, for simplicity, we work in units in which $16\pi G = 1$.

In this exercise is to vary this action with respect to the metric, and show that it gives the Einstein equation $G_{\mu\nu} = 0$.

1. Show that
$$\delta S_{EH} = \int d^4x \left[\delta(\sqrt{-g}g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right] \tag{14}$$

2. Using (3), show that
$$\delta(\sqrt{-g}g^{\mu\nu})R_{\mu\nu} = \sqrt{-g}(\delta g^{\alpha\beta})G_{\alpha\beta} \tag{15}$$
If the contribution from the second term in (14) vanishes, then deduce that one arrives at the required Einstein equation.

3. **Extra** : The second term in (14) is more involved. It will be useful to work with the Riemann tensor, and hence with the definition $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. In locally inertial coordinates, show that
$$\delta R^{\alpha}{}_{\mu\alpha\nu} = \partial_\beta(\delta\Gamma^{\alpha}_{\mu\nu}) - \partial_\nu(\delta\Gamma^{\alpha}_{\mu\beta}) \qquad \text{(locally inertial coordinates)} \tag{16}$$
Now, while Γ is not a tensor, show that $\delta\Gamma$ *is* a tensor. Deduce therefore that in any coordinates
$$\delta R^{\alpha}{}_{\mu\alpha\nu} = \nabla_\beta(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\alpha}_{\mu\beta})$$
This identity is known as the *Palatini identity*. Deduce that
$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha \mathcal{U}^\alpha$$
where the vector density $\mathcal{U}^\alpha$ is given by
$$\mathcal{U}^\alpha = \sqrt{-g}(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\beta}_{\mu\beta}) \tag{17}$$
Using (4), show therefore that
$$\delta S_{EH} = \int d^4x \left[\sqrt{-g}(\delta g^{\alpha\beta})G_{\alpha\beta} + \partial_\alpha \mathcal{U}^\alpha\right]$$
Finally, using the divergence theorem, the last term can be written as an integral over the boundary of the space-time manifold, namely $\int d^3x n_\alpha \mathcal{U}^\alpha$ where n_α is a normal to the boundary. Assuming that the variations of the metric vanish on the boundary, or on considering a space-time with no boundary, this last term will give zero. Hence we arrive
$$\frac{\delta S_{EH}}{\delta g^{\alpha\beta}} = \sqrt{-g}G_{\alpha\beta} \tag{18}$$
and thus minimising the action gives the Einstein equation.

7 Einstein equation with matter

When matter is present, the action giving the Einstein equations is
$$S = S_{EH} + S_{matter} \tag{19}$$

1. On defining
$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left(\frac{\delta S_{matter}}{\delta g^{\mu\nu}}\right)} \tag{20}$$
show, using the results of the previous exercise, that variation of the action S yields $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

2. Now we consider matter consisting of a free massive scalar field, ϕ described by the Klein-Gordon action which you have seen in your Quantum Field theory course (however, note are sign differences because our metric has the opposite sign to that of QFT!) :

$$S_{matter}^{scalar} = \frac{1}{2}\int d^4x\sqrt{-g}\left(-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2\right) \tag{21}$$

Show that in this case

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) + m^2\phi^2) \tag{22}$$

Is this $T_{\mu\nu}$ symmetric? Show explicitly that it is conserved using the equation of motion for ϕ. Deduce the energy density ρ and pressure P of this massive free scalar field. How do these expressions fpr ρ and P simplify when the scalar field depends only on time, namely $\phi = \phi(t)$? (This is a situation we will meet in the future)?

3. When matter consists of a *cosmological constant*,

$$S_{matter}^{CC} = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\Lambda \tag{23}$$

Calculate $T_{\mu\nu}$. Why must Λ be a constant?