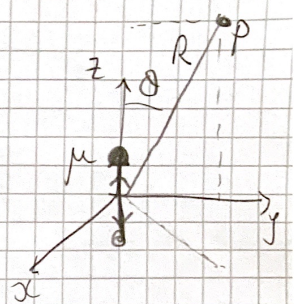
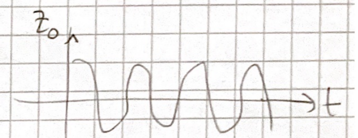
# $\boxed{\text{Ex3 from TD4}}$




GWs from non-relativic system,
mass $\mu$ oscillating along $z$ axis:

$$z_0(t) = A \cos \omega_s t$$




(i) $\rho = \mu \delta(x)\, \delta(y)\, \delta(z - z_0(t))$

$$I_{ij} = \int d^3y \; T^{00} \, y_i y_j \qquad \text{where } \vec{y} = (x, y, z).$$

Only non-zero component

$$I_{zz} = \mu \, z_0^2(t)$$

(ii) Want $h_{ij}^{TT}$ at point P on figure above

$$\dot{I}_{zz} = 2\mu z_0 \dot{z}_0$$

$$\ddot{I}_{zz} = 2\mu\left(\dot{z}_0^2 + z_0 \ddot{z}_0\right)$$

$$= 2\mu A^2 \left((\omega_s \sin \omega_s t)^2 + \cos \omega_s t \left(-\omega_s^2 \cos \omega_s t\right)\right)$$

$$= 2\mu A^2 \omega_s^2 \left(\sin^2 \omega_s t - \cos^2 \omega_s t\right)$$

$$= -2\mu A^2 \omega_s^2 \cos(2\omega_s t)$$

From the formula sheet, $\bar{h}_{ij}(t, \underbrace{\vec{x}}_{=\vec{n}R}) = \dfrac{2G}{c^4 R} \ddot{I}_{ij} \Big|_{t_r = t - R}$

$$\Rightarrow \bar{h}_{ij}(t, \vec{x}) = \overbrace{\mp \frac{4G\mu}{c^4 R} (A\omega_s)^2 \cos(2\omega_s(t-R))}^{F} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{TT} \; — \; *$$

This is clearly neither transverse nor traceless

$$\bar{h}_{ij}(R\, n^j) \neq 0 \qquad\qquad \text{trace} = -1$$

Here $\vec{n} = (0, \sin\theta, \cos\theta)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \neq 0.$$

- So now need to extract the TT component from ⊛

- Two methods.
  a) construct the projection tensor.
  b) more clever one proposed in the exercise.

Lets try a) (mainly because I never tried that!)

$$P_{ij} = \delta_{ij} - \underbrace{n_i n_j}$$

$n_1 = 0$
$n_2 = s$ (= sin)
$n_3 = c$ (→ cos)

$$M_{ij} = n_i n_j = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & s^2 & sc \\ 0 & sc & c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-s^2 & -sc \\ 0 & -sc & 1-c^2 \end{pmatrix} \qquad c^2 + s^2 = 1$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 & -sc \\ 0 & -sc & s^2 \end{pmatrix} \qquad \text{symm.}$$

$$P n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 & -sc \\ 0 & -sc & s^2 \end{pmatrix}\begin{pmatrix} 0 \\ s \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c^2(s) - sc^2 \\ 0 \end{pmatrix} = 0 \checkmark$$

From formula sheet:

$$h_{ij}^{TT} = \left( P_i{}^k \bar{h}_{kl} P^l{}_j \right) - \frac{1}{2} P_{ij} \left[ P^{lk} \bar{h}_{kl} \right] \qquad \text{—— ①}$$

Let's do each line separately. The $2^{nd}$ looks easier.
But first lets write things in matrix form:

$\underline{\underline{h}}^{TT}$ is matrix with components $h_{ij}^{TT}$

$\underline{\underline{\bar{h}}}$ " " " " $\bar{h}_{ij}$

$\underline{\underline{P}}$ " " " " $P_{ij}$

So ① is in matrix form:

$$\underline{\underline{h}}^{TT} = \left( \underline{\underline{P}}\, \underline{\underline{\bar{h}}}\, \underline{\underline{P}} \right) - \frac{1}{2} \underline{\underline{P}}\, \mathrm{Tr}\left( \underline{\underline{P}}\, \underline{\underline{\bar{h}}} \right). \qquad \text{—— ②}$$

amplitude in ⊛

$2^{nd}$ term:

$$\underline{\underline{P}}\,\underline{\underline{\bar{h}}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 & -sc \\ 0 & -sc & s^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & sc \\ 0 & 0 & -s^2 \end{pmatrix} F$$

$$\Rightarrow \mathrm{Tr}\left( \underline{\underline{P}}\,\underline{\underline{\bar{h}}} \right) = -s^2$$

1st term in 2

$$\underline{\underline{P}}\,\underline{\underline{h}}\,\underline{\underline{P}} = F\begin{pmatrix}1 & 0 & 0\\ 0 & c^2 & -sc\\ 0 & -sc & s^2\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & c^2 & -sc\\ 0 & -sc & s^2\end{pmatrix}$$

$$= F\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & sc\\ 0 & 0 & -s^2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & c^2 & -sc\\ 0 & -sc & s^2\end{pmatrix}$$

$$= F\begin{pmatrix}0 & 0 & 0\\ 0 & -(sc)^2 & s^2(sc)\\ 0 & s^2(sc) & -s^4\end{pmatrix}.$$

$$\Rightarrow \underline{\underline{h}}^{TT} = F\left[s^2\begin{pmatrix}0 & 0 & 0\\ 0 & -c^2 & sc\\ 0 & sc & -s^2\end{pmatrix} + \frac{s^2}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & c^2 & -sc\\ 0 & -sc & s^2\end{pmatrix}\right]$$

$$= F\frac{s^2}{2}\left[\begin{pmatrix}0 & 0 & 0\\ 0 & -2c^2 & 2sc\\ 0 & 2sc & -2s^2\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & c^2 & -sc\\ 0 & -sc & s^2\end{pmatrix}\right]$$

$$= F\frac{s^2}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & -c^2 & +sc\\ 0 & sc & -s^2\end{bmatrix}$$

answer

no one got here

Which has zero trace and is $\perp$ to $\hat{n}$ ✓

In fact, in exercise 1 gave the answer in a coord system in which wave propagating along $\vec{n}$, so rotated by an angle $\theta$ around the $\vec{x}$ axis. 5

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}.$$

$$h'^{TT}_{ij} = \underline{\underline{R}}^{-1} \, \underline{\underline{h}}^{TT} \, \underline{\underline{R}}$$

$$\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & +s & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c^2 & sc \\ 0 & sc & -s^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & +s \\ 0 & -s & c \end{pmatrix}$$

$$\propto \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c^3 - s^2c & -c^2s + sc^2 \\ 0 & sc^2 + s^3 & +s^2c - s^2c \end{pmatrix}$$

$$\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & +s & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c & 0 \\ 0 & s & 0 \end{pmatrix}$$

$$\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c^2 - s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ as asked.