TD2

The solution of Ex 3 may be found in the book of D.Langlois.

Comment

(1) * $\nabla_\kappa \nabla_\nu V_\mu - \nabla_\nu \nabla_\kappa V_\mu = - V_\sigma R^\sigma{}_{\mu\nu\kappa}$

* $$\nabla_\kappa \nabla_\nu V^\mu - \nabla_\nu \nabla_\kappa V^\mu = \nabla_\kappa \nabla_\nu (g^{\mu\alpha} V_\alpha) - \nabla_\nu \nabla_\kappa (g^{\mu\alpha} V_\alpha)$$
$$= g^{\mu\alpha} \left(\nabla_{[\kappa} \nabla_{\nu]} V_\alpha \right)$$
$$= g^{\mu\alpha} \left(- V_\sigma R^\sigma{}_{\alpha\nu\kappa} \right)$$
$$= - V_\sigma R^{\sigma\mu}{}_{\nu\kappa}$$
$$= - V^\sigma R_\sigma{}^\mu{}_{\nu\kappa}$$
$$= + V^\sigma R^\mu{}_{\sigma\nu\kappa}$$

using antisym props. of Riemann.

(2) * $\nabla_\gamma g_{\alpha\beta} = 0$ (follows from definition of the covariant deriv).

Since $g_{\alpha\beta} g^{\beta\delta} = \delta^\delta{}_\beta$

$$\Rightarrow \quad \nabla_\gamma (\delta^\delta{}_\beta) = 0 = \nabla_\gamma (g_{\alpha\beta} g^{\beta\delta}) = (\nabla_\gamma g_{\alpha\beta}) g^{\beta\delta} + g_{\alpha\beta} (\nabla_\gamma g^{\beta\delta}) = 0$$

$\Rightarrow$ Since $\nabla_\gamma g_{\alpha\beta} = 0 \quad \Rightarrow \quad \nabla_\gamma g^{\beta\delta} = 0$

Ex 2

1) $M'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} M_{\gamma\delta}$ (1)

So in matrix notation, letting $\underline{\underline{M}}$ be 4×4 matrix with cpts $M_{\alpha\beta}$

$\underline{\underline{M}}'$ . . . . . $M'_{\alpha\beta}$

$\underline{\underline{J}}$ . . . . $\frac{\partial x^\alpha}{\partial x'^\beta}$

This is $\underline{\underline{M}}'_{\alpha\beta} = \left[\underline{\underline{J}}\,\underline{\underline{M}}\,\underline{\underline{J}} \right]_{\alpha\beta}$

$\Rightarrow \underline{\underline{M}}' = \underline{\underline{J}}\,\underline{\underline{M}}\,\underline{\underline{J}}$

$\Rightarrow \det \underline{\underline{M}}' = \det \underline{\underline{J}} \det \underline{\underline{M}} \det \underline{\underline{J}}$

$\Rightarrow (\det \underline{\underline{M}}') = (\det \underline{\underline{M}}) (\det \underline{\underline{J}})^2$

$\Rightarrow \sqrt{\det M'} = (\det J) \sqrt{\det M}$ $\Rightarrow$ $\sqrt{\det M'}$ is a scalar density.

2) $\Rightarrow \sqrt{-g} = \sqrt{-(\det g_{\alpha\beta})}$ is a scalar density.

need − as det is −ve

3) S scalar as E eq^{n}s must be true ∀ coord system. They come from minimising S, so S had better be a scalar...

$S = \int d^4x\, \mathcal{L} = \int d^4x'\, \mathcal{L}'$ (as S a scalar)

$= \int d^4x' \, \|J\| \, \mathcal{L}$

$\Rightarrow \mathcal{L}' = \|J\| \mathcal{L} = \left| \frac{\partial x}{\partial x'} \right| \mathcal{L}$ so $\mathcal{L}$ is a scalar density.

· So $\mathcal{L}' = \sqrt{-g}\, \Phi$, with Φ a scalar, is a scalar density $\Rightarrow$ $\sqrt{-g(x)}\, d^4x\, \Phi = \sqrt{-g(x')}\, d^4x'\, \Phi$

(as $\Phi' = \Phi$) $\Rightarrow$ $\sqrt{-g}\, d^4x = \sqrt{-g'}\, d^4x'$

= invariant measure

5) $$\det \underline{\underline{M}} = e^{\mathrm{Tr}(\ln \underline{\underline{M}})}$$

$$\Rightarrow \delta(\det \underline{\underline{M}}) = \mathrm{Tr}\left(\underline{\underline{M}}^{-1} \delta \underline{\underline{M}}\right) e^{\mathrm{Tr}(\ln \underline{\underline{M}})}$$

$$= \left(M^{\alpha\beta} \delta M_{\alpha\beta}\right) \det M$$

$$\Rightarrow \delta g = \left(g^{\alpha\beta} \delta g_{\alpha\beta}\right) g = - g \, g_{\alpha\beta} \, \delta g^{\alpha\beta}$$ [since $g^{\alpha\beta} g_{\beta\omega} = \delta^{\alpha}_{\omega}$]

$$\Rightarrow \delta\sqrt{-g} = \frac{1}{2} \frac{1}{\sqrt{-g}} (-\delta g)$$

$$= \frac{1}{2} \frac{1}{\sqrt{-g}} (-g) \, g^{\alpha\beta} \delta g_{\alpha\beta}$$

$$= \frac{1}{2} \sqrt{-g} \, g^{\alpha\beta} \delta g_{\alpha\beta} = -\frac{1}{2} \sqrt{-g} \, g_{\alpha\beta} \delta g^{\alpha\beta}.$$

6) $$\partial_\mu \left(\sqrt{-g}\right) = \frac{1}{2} \sqrt{-g} \, g^{\alpha\beta} g_{\alpha\beta,\mu}$$

$$\& \quad \Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} g^{\alpha\varepsilon} \left(\cancel{g_{\varepsilon\mu,\alpha}} + g_{\varepsilon\alpha,\mu} - \cancel{g_{\mu\alpha,\varepsilon}} \right) = \frac{1}{2} g^{\alpha\varepsilon} g_{\varepsilon\alpha,\mu}$$

$$\Rightarrow \partial_\mu \left(\sqrt{-g}\right) = \sqrt{-g} \, \Gamma^{\alpha}_{\mu\alpha}$$

Then what...

$$\nabla_\alpha g^{\mu\nu} = 0 = \partial_\alpha g^{\mu\nu} + \Gamma^{\mu}_{\varepsilon\alpha} g^{\varepsilon\nu} + \Gamma^{\nu}_{\varepsilon\alpha} g^{\varepsilon\mu}$$

$$\Rightarrow \partial_\alpha g^{\mu\nu} = - \Gamma^{\mu}_{\varepsilon\alpha} g^{\varepsilon\nu} - \Gamma^{\nu}_{\varepsilon\alpha} g^{\varepsilon\mu}$$

$$\Rightarrow \partial_\beta \left(\sqrt{-g} \, g^{\alpha\beta}\right) = \left(\partial_\beta \sqrt{-g}\right) g^{\alpha\beta} + \left(\partial_\beta g^{\alpha\beta}\right) \sqrt{-g}$$

$\alpha \to \beta$
$\mu \to \alpha$
$\nu \to \beta$ ~~β~~

$$= \sqrt{-g} \left[\cancel{g^{\alpha\beta} \Gamma^{\omega}_{\beta\omega}} - \left(\Gamma^{\alpha}_{\varepsilon\nu} g^{\varepsilon\nu} + \cancel{\Gamma^{\beta}_{\varepsilon\beta} g^{\varepsilon\alpha}} \right) \right]$$

$$= - \sqrt{-g} \, \Gamma^{\alpha}_{\varepsilon\nu} g^{\varepsilon\nu}$$

7) $$\nabla_\alpha \mathcal{A} = \nabla_\alpha\left(\sqrt{-g}\,A\right) = \sqrt{-g}\left(\nabla_\alpha A\right) = \sqrt{-g}\left(\partial_\alpha A\right)$$

and $\partial_\alpha\left(\sqrt{-g}\,A\right)$

$$= \left(\partial_\alpha \sqrt{-g}\right)A + \sqrt{-g}\left(\partial_\alpha A\right) = \sqrt{-g}\,\Gamma^\mu_{\alpha\mu} A + \sqrt{-g}\left(\partial_\alpha A\right)$$

$$\Rightarrow \quad \partial_\alpha \mathcal{A} = \Gamma^\mu_{\alpha\mu}\mathcal{A} + \nabla_\alpha \mathcal{A}$$

$$\Rightarrow \quad \boxed{\nabla_\alpha \mathcal{A} = \partial_\alpha \mathcal{A} - \Gamma^\mu_{\alpha\mu}\mathcal{A}}$$

8) same idea...

9) from result of part 8 follows 4

$$\nabla_\alpha \mathcal{V}^\alpha = \partial_\alpha \mathcal{V}^\alpha \quad \text{— (a)}$$

$$\Rightarrow \quad \nabla_\alpha\left(\sqrt{-g}\,V^\alpha\right) = \sqrt{-g}\left(\nabla_\alpha V^\alpha\right) \overset{(a)}{=} \partial_\alpha\left(\mathcal{V}^\alpha\right) = \partial_\alpha\left(\sqrt{-g}\,V^\alpha\right)$$

$$\Rightarrow \quad \boxed{\nabla_\alpha V^\alpha = \frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}\,V^\alpha\right)}$$

The solution of Ex 3 may be found in the book of D.Langlois.

TD2 Ex4

- $x^\alpha = r^\alpha(\tau, \mu)$

μ: param labelling different geodesics

τ: param along geodesic, ie proper-time for time-like geodesics

$$u^\mu = \frac{\partial r^\mu}{\partial \tau}, \qquad \xi^\mu = \frac{\partial r^\mu}{\partial \mu}$$

1) geodesic eq^{n} $u^\beta \nabla_\beta u^\alpha = 0$ as seen in lectures.

2)
$$\xi^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta \xi^\alpha$$

$$= \frac{\partial r^\beta}{\partial \mu} \partial_\beta \left(\frac{\partial r^\alpha}{\partial \tau}\right) - \frac{\partial r^\beta}{\partial \tau} \partial_\beta \left(\frac{\partial r^\alpha}{\partial \mu}\right)$$

using chain rule
$$= \frac{\partial}{\partial \mu}\left(\frac{\partial r^\alpha}{\partial \tau}\right) - \frac{\partial}{\partial \tau}\left(\frac{\partial r^\alpha}{\partial \mu}\right) = 0.$$

By exploiting the symm properties of the Christoffel symbols under $\beta \leftrightarrow \gamma$

$$\xi^\beta \nabla_\beta u^\alpha - u^\beta \nabla_\beta \xi^\alpha$$

$$= \left(\xi^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta \xi^\alpha\right) + \underbrace{\left(\xi^\beta \Gamma^\alpha_{\beta\gamma} u^\gamma - u^\beta \Gamma^\alpha_{\gamma\beta} \xi^\gamma\right)}_{0}$$

$$= \xi^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta \xi^\alpha$$

So we arrive at $\boxed{\xi^\beta \nabla_\beta u^\alpha - u^\beta \nabla_\beta \xi^\alpha = 0}$ — (*)

3). Now, use that

$$\nabla_\alpha \nabla_\beta A^\mu - \nabla_\beta \nabla_\alpha A^\mu = R^\mu{}_{\nu\alpha\beta} A^\nu$$

(see eg first set of TD solutions handed out, page 2)

so

$$\nabla_\gamma \nabla_\delta u^\alpha - \nabla_\delta \nabla_\gamma u^\alpha = R^\alpha{}_{\nu\gamma\delta} u^\nu$$

& hence

$$\boxed{\xi^\gamma u^\delta \left(\nabla_\gamma \nabla_\delta u^\alpha - \nabla_\delta \nabla_\gamma u^\alpha\right) = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\delta \xi^\gamma}$$

as required — (**)

4) Now 1st term above is,

$$\underbrace{\xi^\gamma u^\delta \nabla_\gamma\left(\nabla_\delta u^\alpha\right)}_{} = \xi^\gamma \nabla_\gamma \underbrace{\left[u^\delta \nabla_\delta u^\alpha\right]}_{=0 \text{ by geodesic eq}^n} - \xi^\gamma \left(\nabla_\gamma u^\delta\right)\left(\nabla_\delta u^\alpha\right)$$

$$= -\xi^\gamma \left(\nabla_\gamma u^\delta\right)\left(\nabla_\delta u^\alpha\right)$$

& 2nd term, in a similar way

$$\underbrace{\xi^\gamma u^\delta \nabla_\delta\left(\nabla_\gamma u^\alpha\right)}_{} = u^\delta \nabla_\delta \underbrace{\left[\xi^\gamma \nabla_\gamma u^\alpha\right]}_{\substack{\text{using } (*) \\ = u^\gamma \nabla_\gamma \xi^\alpha}} - u^\delta \left(\nabla_\delta \xi^\gamma\right)\left(\nabla_\gamma u^\alpha\right)$$

$$= u^\delta \nabla_\delta \left(u^\gamma \nabla_\gamma \xi^\alpha\right) - u^\delta \left(\nabla_\delta \xi^\gamma\right)\left(\nabla_\gamma u^\alpha\right)$$

Hence (**) is

$$R^{\alpha}{}_{\beta\gamma\delta}\, u^{\beta} u^{\delta} \xi^{\gamma} = -\xi^{\epsilon}\left(\nabla_{\gamma} u^{\delta}\right)\left(\nabla_{\delta} u^{\alpha}\right) + - u^{\delta} \nabla_{\delta}\left(u^{\gamma} \nabla_{\gamma} \xi^{\alpha}\right) + u^{\delta}\left(\nabla_{\delta} \xi^{\gamma}\right)\left(\nabla_{\gamma} u^{\alpha}\right)$$

$$= -u^{\delta} \nabla_{\delta}\left(u^{\gamma} \nabla_{\gamma} \xi^{\alpha}\right) + \left(\nabla_{\gamma} u^{\alpha}\right)\left[u^{\delta}\left(\nabla_{\delta} \xi^{\gamma}\right) - \xi^{\delta}\left(\nabla_{\delta} u^{\gamma}\right)\right]$$

so we arrive at

$$\boxed{-R^{\alpha}{}_{\beta\gamma\delta}\, u^{\beta} u^{\delta} \xi^{\gamma} = u^{\delta} \nabla_{\delta}\left(u^{\gamma} \nabla_{\gamma} \xi^{\alpha}\right) - \left(\nabla_{\gamma} u^{\alpha}\right)\left[u^{\delta}\left(\nabla_{\delta} \xi^{\gamma}\right) - \xi^{\delta}\left(\nabla_{\delta} u^{\gamma}\right)\right]}$$

this vanishes by virtue of (*)

$$\Rightarrow \boxed{u^{\delta} \nabla_{\delta}\left(u^{\gamma} \nabla_{\gamma} \xi^{\alpha}\right) = -R^{\alpha}{}_{\beta\gamma\delta}\, u^{\beta} u^{\delta} \xi^{\gamma}}$$

The first term can be written in terms of

$\frac{D^2 \xi^{\alpha}}{D\tau^2}$, since

$$\frac{D\xi^{\alpha}}{D\tau} = \frac{d\xi^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\mu\nu} \xi^{\mu} u^{\nu} \;\hat{=}\; u^{\gamma} \nabla_{\gamma} \xi^{\alpha}$$

So
$$u^\delta \nabla_\delta \left(u^\gamma \nabla_\gamma \xi^\alpha \right)$$

$$= u^\delta \nabla_\delta \left[\frac{D\xi^\alpha}{D\tau} \right]$$

$$= \frac{D}{D\tau} \left[\frac{D\xi^\alpha}{D\tau} \right]$$

$$= \frac{D^2\xi^\alpha}{D\tau^2}.$$

Thus we arrive at (7):

$$\boxed{\frac{D^2\xi^\alpha}{D\tau^2} = - R^\alpha{}_{\beta\gamma\delta} \, u^\beta \xi^\gamma u^\delta}$$

TD 2 ex5

1) Have $\nabla_{[\kappa} \nabla_{\nu]} V_\mu = -V_\sigma R^\sigma{}_{\mu\nu\kappa}$

so $$[\nabla_b \nabla_c - \nabla_c \nabla_b] V_d = -V_f R^f{}_{dcb}$$

Hence $$\nabla_a\big((\nabla_b \nabla_c - \nabla_c \nabla_b) V_d\big)$$

$$= - \nabla_a \left(V_f R^f{}_{dcb}\right)$$

$$= -(\nabla_a V_f) R^f{}_{dcb} - V_f \left(\nabla_a R^f{}_{dcb}\right)$$

$= R_{cb}{}^f{}_d$

$= R_{bcd}{}^f$

2). for a tensor, T_{cd},

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T_{cd} = -T_{cf} R^f{}_{dba} - T_{fd} R^f{}_{cba}$$

since $\nabla_c V_d$ is a tensor, by construction,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(\nabla_c V_d) = -R^f{}_{dba}(\nabla_c V_f)$$
$$- R^f{}_{cba}(\nabla_f V_d)$$

now a question of signs...

$$- R^f{}_{dba} \nabla_c V_f = -R_{ba}{}^f{}_d \nabla_c V_f$$
$$= + R_{bad}{}^f \nabla_c V_f$$

$$= - R_{abd}{}^{f} \nabla_c V_f$$ hmm...

& $$- R^f{}_{cba}(\nabla_f V_d) = - R_{ba}{}^{e}{}_{c}(\nabla_e V_d)$$
$$= + R_{bac}{}^{e}(\nabla_e V_d)$$
$$= - R_{abc}{}^{e}(\nabla_e V_d)$$

Let me stick with signs here, which I think are correct (error in typing the TD, sorry...)

So $$(\nabla_a \nabla_b - \nabla_b \nabla_a)(\nabla_c V_d) = - R_{abc}{}^{e}(\nabla_e V_d) - R_{abd}{}^{f}(\nabla_c V_f)$$

3) So we have on one hand

$$\nabla_a \nabla_b \nabla_c V_d - \nabla_a \nabla_c \nabla_b V_d \qquad \text{(part 1)}$$

& on the other hand

$$\nabla_a \nabla_b \nabla_c V_d - \nabla_b \nabla_a \nabla_c V_d \qquad \text{(part 2)}$$

1st two terms in each case are the same, so will also be same if antisymmetrize.

last 2 terms differ. But

$$\nabla_{[a} \nabla_c \nabla_{b]}$$

= anti sym under interchange of any 2 indices

so $= - \nabla_{[a} \nabla_b \nabla_{c]}$

$= + \nabla_{[b} \nabla_a \nabla_{c]}$

Hence when antisymmized, parts 1 & 2 are identical.

$$\nabla_{[a} \left(\nabla_b \nabla_{c]} - \nabla_c \nabla_{b]} \right) V_d$$

$$= \nabla_{[a} \nabla_b - \nabla_b \nabla_a) \nabla_{c]} V_d$$

$$\Rightarrow \; - \underline{\nabla_{[a} V_{|f|} R_{bc]d}{}^f} - V_f \nabla_{[a} R_{bc]d}{}^f$$

$$= - \cancel{R_{[abc]}{}^e} \nabla_e V_d - \underline{R_{[ab|d|}{}^f \nabla_{c]} V_f}$$

which is (23).

- Now, from the cyclicity (cond^n) seen in lectures,

$$R_{[abc]}{}^e = 0.$$

- The terms underlined are identical as $[abc] = [bca]$

So we arrive at

$$V_f \nabla_{[a} R_{bc]d}{}^f = 0 .$$

- This is true $\forall \; V_f \Rightarrow \boxed{\nabla_{[a} R_{bc]d}{}^f = 0}$

Ex 6

$$S_{EH} = \int d^4x \sqrt{-g}\, R = \int d^4x \sqrt{-g}\,(g^{\mu\nu} R_{\mu\nu})$$

2)

$$\delta(\sqrt{-g}\, g^{\mu\nu}) = \delta(\sqrt{-g})\, g^{\mu\nu} + \sqrt{-g}\, \delta g^{\mu\nu}$$

from eq(3) $\rightarrow$

$$= -\tfrac{1}{2}\sqrt{-g}\,(g_{\alpha\beta}\, \delta g^{\alpha\beta})\, g^{\mu\nu} + \sqrt{-g}\, \delta g^{\mu\nu}$$

$$\therefore \quad R_{\mu\nu}\, \delta(\sqrt{-g}\, g^{\mu\nu}) = -\tfrac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} R + \sqrt{-g}(\delta g^{\mu\nu})\, R_{\mu\nu}]$$

$$= \delta g^{\mu\nu} \sqrt{-g} \left[R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right]$$

$$= \delta g^{\mu\nu} \sqrt{-g}\, G_{\mu\nu}.$$

1) 3) So

$$\delta S_{EH} = \int d^4x \left[\delta g^{\mu\nu} \sqrt{-g}\, G_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} \right].$$

Let's look at 2nd term now.

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

$$\& \quad R^{\alpha}{}_{\mu\beta\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma\Gamma - \Gamma\Gamma.$$

In locally Minkowski coords $\Gamma = 0$, $\partial\Gamma \neq 0$, so in these coords

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \partial_\alpha(\delta\Gamma^{\alpha}_{\mu\nu}) - \partial_\nu(\delta\Gamma^{\alpha}_{\mu\beta}) \quad — ①$$

Now, $\delta\Gamma^{\alpha}_{\mu\beta}$ ... Recall transf^n law for Γ was (neglecting all indices!)

$$\Gamma' = \frac{\partial x'}{\partial x}\frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'}\Gamma + \frac{\partial x'}{\partial x}\frac{\partial^2 x}{\partial x' \partial x'}$$

This last term is indep of Γ, so

$$\delta\Gamma' = (\quad\downarrow\quad)\,\delta\Gamma \quad \Rightarrow \quad \delta\Gamma^{\alpha}_{\beta\gamma} \text{ is a tensor}$$

~~Since~~ Then ① can be rewritten as

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \nabla_\alpha(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\alpha}_{\mu\beta})$$

and must be true $\forall$ coord as a tensor eq^n.

This is the Palatini identity. Contracting it therefore gives

$$\Rightarrow \quad \delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha})$$

So $\sqrt{-g}\, g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\alpha \mathcal{U}^\alpha$ where $\mathcal{U}^\alpha$ ~~g~~ is what is written in (12)

which is a ~~tensor~~ vector density

" $\mathcal{U} = \sqrt{-g}\,(g\,\delta\Gamma - g\,\delta\Gamma)$ "

and since $\nabla_\alpha \mathcal{U}^\alpha = \partial_\alpha \mathcal{U}^\alpha$ (as shown in (4))

$$\Rightarrow \quad \delta S_{EH} = \int d^4x \left(\delta g^{\alpha\beta} G_{\alpha\beta}\sqrt{-g} + \partial_\alpha \mathcal{U}^\alpha\right)$$

last term $\int d^4x\,(\partial_\alpha \mathcal{U}^\alpha) = \int_{\text{surface}} d^3x\; n_\alpha \mathcal{U}^\alpha$

where the surface = boundary of space-time, with normal n_α.

↑ used in space-times with boundaries

TD2, ex7

1) $$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left(\frac{\delta S_M}{\delta g^{\mu\nu}}\right)$$

$$S = S_{EH} + S_M$$

$$\delta S = 0 \Rightarrow \delta S_{EH} = -\delta S_M$$

$$\Rightarrow \cancel{\sqrt{-g}}\, G_{\mu\nu} = -\left(-\frac{\cancel{\sqrt{-g}}}{2} T_{\mu\nu}\right)$$

$$G_{\mu\nu} = \frac{T_{\mu\nu}}{2}$$

since have set "$16\pi G = 1$", left with factor of $8\pi G$.

2) $$S^{\phi}_{matter} = \frac{1}{2}\int d^4x \sqrt{-g}\left(-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2\right)$$

$\frac{\delta S^{\phi}_{matter}}{\delta\phi} = 0$ gives e of m for scalar field

$\frac{\delta S^{\phi}_{matter}}{\delta g^{\mu\nu}}$ will give stress energy tensor

} in fact in this part we will need both

Let's vary wrt metric first (with ϕ fixed)

$$\delta_g S^{\phi}_{matter} = \frac{1}{2}\int d^4x \left(\left(\delta\sqrt{-g}\right)\left(-(\partial\phi)^2 - m^2\phi^2\right) + \sqrt{-g}\left(-\delta g^{\mu\nu}\right)\left(\partial_\mu\phi\,\partial_\nu\phi\right)\right)$$

$= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$, see ex 2

$$\delta_g S^{\phi}_{matter} = \frac{1}{2}\int d^4x \sqrt{-g}\left[\frac{1}{2} g_{\mu\nu}\left((\partial\phi)^2 + m^2\phi^2\right) - \partial_\mu\phi\,\partial_\nu\phi\right]\delta g^{\mu\nu} \quad (2)$$

$$\Rightarrow \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{\phi}_{matter}}{\delta g^{\mu\nu}} = -\frac{\cancel{2}}{\cancel{\sqrt{-g}}}\,\cancel{\frac{1}{2}}\,\cancel{\sqrt{-g}}\left[\qquad\right]$$

$$\boxed{T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + m^2\phi^2\right)}$$

Clearly yes, $T_{\mu\nu} = T_{\nu\mu}$.

- Now want e of m for ϕ, so $\delta_\phi S = 0$

vary wrt ϕ with $g_{\mu\nu}$ fixed

deriv of a product, + use symm.

$$\delta_\phi S = \cancel{\tfrac{1}{2}}\int d^4x\sqrt{-g}\left(-\cancel{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu(\delta\phi)) - \cancel{2}m^2\phi\,\delta\phi\right)$$

integ by parts

$$= +\int d^4x\left(\partial_\nu\left[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right]\delta\phi - \sqrt{-g}\,m^2\phi\,\delta\phi\right)$$

+ boundary term I will neglect assuming $\delta\phi = 0$ on bdry, i.e. ϕ fixed on bdry.

$$\delta_\phi S = 0 \quad \Rightarrow \frac{1}{\sqrt{-g}}\partial_\nu\underbrace{\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)}_{\text{vector density}} - m^2\phi = 0$$

$\to$ from ex 2, part 9 after eq (4)

$$\nabla_\mu\left(g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0$$

to

$$\nabla_\mu\left(\partial^\mu\phi\right) = m^2\phi\,,$$

& equivalently $\boxed{\nabla_\mu\nabla^\mu\phi = m^2\phi}$

since $\partial^\mu\phi = \nabla^\mu\phi$ as ϕ is a scalar.

using (**) there are 3 terms

Now want to show that $\nabla_\mu T^{\mu\nu} = 0$

$$\nabla_\mu\left(\partial^\mu\phi\,\partial^\nu\phi\right) = \left[\nabla_\mu\left(\nabla^\mu\phi\right)\right]\partial^\nu\phi + \left(\partial^\mu\phi\right)\nabla_\mu\partial^\nu\phi$$

$$= m^2\phi\,\partial^\nu\phi + \left(\partial^\mu\phi\right)\left(\nabla_\mu\nabla^\nu\phi\right)$$

& $$\nabla_\mu\left[g^{\mu\nu}\left(\partial_\alpha\phi\right)^2\right] = g^{\mu\nu}\,\nabla_\mu\left[\partial_\alpha\phi\,\partial_\beta\phi\,g^{\alpha\beta}\right]$$

$$= 2g^{\mu\nu}\left[\nabla_\mu\left(\partial_\alpha\phi\right)\,\partial^\alpha\phi\right]$$

& $$\nabla_\mu\left[m^2\phi^2 g^{\mu\nu}\right] = 2m^2\phi\left(\nabla_\mu\phi\right)g^{\mu\nu} = 2m^2\phi\left(\nabla^\nu\phi\right)$$

So, $$\nabla_\mu T^{\mu\nu} = m^2\phi\,\partial^\nu\phi + \left(\partial^\mu\phi\right)\left(\nabla_\mu\nabla^\nu\phi\right)$$

$$-\frac{1}{\cancel{2}}\left\{\cancel{2}g^{\mu\nu}\nabla_\mu\left(\nabla_\alpha\phi\right)\partial^\alpha\phi + \cancel{2}m^2\phi\,\nabla^\nu\phi\right\}.$$

$$= 0 \quad \checkmark.$$

· Now, energy density & pressure.

general def^n was $T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu}$ — (***)

with $u^2 = -1$

from (**) we read off that

$$\boxed{P = -\tfrac{1}{2}\left((\partial_\alpha \phi)^2 + m^2 \phi^2 \right)}$$

& $u_\mu = N\, \partial_\mu \phi$ where N = norm. const such that $u^2 = -1$

ie $N^2 (\partial_\mu \phi)(\partial^\mu \phi) = -1$

$$N = \frac{\pm 1}{\sqrt{-(\partial_\mu \phi)^2}}.$$

Thus (***) is

$$T_{\mu\nu} = (\rho + P)\, N^2\, \partial_\mu \phi\, \partial_\nu \phi + P\, g_{\mu\nu}$$

so comparing with (**) $(\rho + P)\, N^2 = 1$

$$\Rightarrow \quad \rho + P = 1/N^2 = -(\partial_\mu \phi)^2$$

Hence $\rho = -(\partial_\mu \phi)^2 - P =$

$$= -(\partial_\mu \phi)^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m^2 \phi^2$$

$$\boxed{\rho = -\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m^2 \phi^2}$$

- when $\phi = \phi(t)$ only,

$$(\partial_\alpha \phi)^2 = -(\partial_t \phi)^2 g^{tt}$$

$$\Rightarrow \rho = \tfrac{1}{2} g^{tt} (\dot{\phi})^2 + \tfrac{1}{2} m^2 \phi^2 \qquad \text{(like KE + PE)}$$

$$P = +\tfrac{1}{2} g^{tt} \dot{\phi}^2 - \frac{m^2 \phi^2}{2}$$

If $g^{tt} = 1$, as in cosmology, $\rho = \frac{\dot{\phi}^2}{2} + \tfrac{1}{2} m^2 \phi^2$

$P = \frac{\dot{\phi}^2}{2} - \tfrac{1}{2} m^2 \phi^2.$

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(3) Cosmological constant.

$$S_{\text{matter}} \propto \int d^4x \sqrt{-g}\,\Lambda = \underbrace{K}_{= 1/8\pi G} \int d^4x \sqrt{-g}\,\Lambda$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}} = -\frac{\cancel{2}}{\cancel{\sqrt{-g}}} K \Lambda \left(-\tfrac{1}{\cancel{2}} \cancel{\sqrt{-g}}\, g_{\mu\nu} \right)$$

$$= K \Lambda g_{\mu\nu}$$

Since $\nabla_\mu T^{\mu\nu} = 0$, it follows that

$$\nabla_\mu \left(\Lambda g^{\mu\nu} \right) = 0$$

$$\Rightarrow g_{\mu\nu} \left(\nabla^\mu \Lambda \right) = 0$$

$$\Rightarrow \Lambda = \text{const.}$$