TD 3

1) Stress energy tensor

1) $$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\sigma\mu} T^{\sigma\nu} + \Gamma^\nu{}_{\sigma\mu} T^{\mu\sigma}$$

Have seen in TD 2 (first box under eq^n (3)) that

$$\Gamma^\alpha{}_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g})$$

$$\Rightarrow \quad \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\sigma\left(\sqrt{-g}\right)T^{\sigma\nu} + \Gamma^\nu{}_{\sigma\mu} T^{\mu\sigma}$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, T^{\mu\nu}\right) + \Gamma^\nu{}_{\sigma\mu} T^{\mu\sigma}$$

2) Perfect fluid of dust, $P = 0 \Rightarrow T^{\mu\nu} = \rho u^\mu u^\nu$

$$\Rightarrow \nabla_\mu(\rho u^\mu u^\nu) = 0 = \nabla_\mu(\rho u^\mu) u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0 \qquad (1)$$

Contract with u_ν (and use that $u^2 = -1$)

$$\Rightarrow \quad -\nabla_\mu(\rho u^\mu) + \rho u^\mu u_\nu (\nabla_\mu u^\nu) = 0$$

Now since $u^\nu u_\nu = -1$
$\Rightarrow (\nabla_\mu u^\nu) u_\nu = 0$ $\Big\}$ hence $\nabla_\mu(\rho u^\mu) = 0 \qquad (2)$

Substitute (2) into (1) $\Rightarrow \quad u^\mu \nabla_\mu u^\nu = 0 \qquad (3)$
Which is nothing other than the geodesic eq^n.

So observe that $\nabla_\mu T^{\mu\nu} = 0$ is telling you about the dynamics of the fluid.

3) In Minkowski space
Want to show that reproduce usual Newtonian e of m for a fluid (continuity & Euler eq^{n}s)

a) In flat space $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \partial_\mu T^{\mu\nu} = 0$
&

$$T^{\mu\nu} = \left(\rho + P/_{c^2}\right) u^\mu u^\nu + p\,\eta^{\mu\nu}$$

$$\Rightarrow \left(\partial_\mu\left[\left(\rho + P/_{c^2}\right)\right]\right) u^\mu u^\nu + \left(\rho + P/_{c^2}\right)\left[(\partial_\mu u^\mu) u^\nu + u^\mu(\partial_\mu u^\nu)\right]$$
$$+ (\partial_\mu P)\eta^{\mu\nu} = 0 \qquad — (4)$$

Now, from $u^\nu u_\nu = -c^2 \Rightarrow (\partial_\mu u^\nu) u_\nu = 0$

So contract (4) with u_ν

$$\Rightarrow -c^2\left(\partial_\mu\left[\rho + \cancel{P/_{c^2}}\right]\right)u^\mu + c^2\left(\rho + P/_{c^2}\right)\left[-\partial_\mu u^\mu\right] + \cancel{(\partial_\mu P) u^\mu} = 0$$

$$\Rightarrow + (\partial_\mu \rho) u^\mu + \rho(\partial_\mu u^\mu) + P/_{c^2}(\partial_\mu u^\mu) = 0$$

$$\Rightarrow \boxed{\partial_\mu(\rho u^\mu) + \frac{P}{c^2}(\partial_\mu u^\mu) = 0} \quad — (5) \quad \text{relativistic continuity eq}^n$$
$$= (\partial_\mu \rho) u^\mu + \rho(\partial_\mu u^\mu) + P/_{c^2}(\partial_\mu u^\mu) = 0$$

(b) Substitute (5) into (4)! so (4) is

$$u^\nu\left[\underline{(\partial_\mu \rho) u^\mu} + \underline{(\partial_\mu u^\mu)}\left(\rho + \underline{P/_{c^2}}\right)\right] + \partial_\mu\left(P/_{c^2}\right) u^\mu u^\nu$$
$$+ \left(\rho + P/_{c^2}\right) u^\mu (\partial_\mu u^\nu) + (\partial_\mu P)\eta^{\mu\nu} = 0$$

use (5)

$$\Rightarrow \cancel{u^\nu(\partial_\mu u^\mu)\rho} + \underline{\partial_\mu\left(P/_{c^2}\right) u^\mu u^\nu} + \left(\rho + P/_{c^2}\right) u^\mu(\partial_\mu u^\nu)$$
$$- \cancel{\rho(\partial_\mu u^\mu) u^\nu} \qquad + (\partial_\mu P)\eta^{\mu\nu} = 0$$

Spherically symm metrics

- ds^2 can only depend on rotational invariants (t, r, $d\vec{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ & $\vec{x}\cdot d\vec{x} = r\,dr$)

$\Rightarrow$ ds^2 takes form in 4:

$$ds^2 = -C(t,r)\,dt^2 + D(t,r)\,dr^2 + 2E(t,r)\,dr\,dt + F(t,r)\,r^2\,d\Omega^2$$

Aim: show that can set $E = 0$ & $F = 1$ by changes of coord.

1) let $r' = r\,F(t,r)$ which can invert $\to r(t,r')$ so that $C(t,r) \to C'(t,r')$ etc & $dr = \alpha(t,r')\,dt + \beta(t,r')\,dr'$

Hence ds^2 of form

$$\Rightarrow ds^2 = -\tilde{C}(t,r')\,dt^2 + \tilde{D}(t,r')\,dr'^2 + 2\tilde{E}(t,r')\,dr'\,dt + r'^2\,d\Omega^2$$

for some functions $\tilde{C}$, $\tilde{D}$ etc, which we don't need to write explicitly

Let's drop the primes & tilda's $\Rightarrow$

$$ds^2 = -C(t,r)\,dt^2 + D(t,r)\,dr^2 + 2E(t,r)\,dr\,dt + r^2\,d\Omega^2. \qquad (*)$$

2) Let

$$dt' = \eta(t,r)\left[C(t,r)\,dt - E(r,t)\,dr\right]$$

where choose η such that this is an exact differential

ie $$dt' = \frac{\partial t'}{\partial t}\,dt + \frac{\partial t'}{\partial r}\,dr$$

So $\frac{\partial}{\partial r}(\eta C) = \frac{\partial}{\partial t}(-E\eta)$

In principle can integrate this to find $\eta(t,r)$ given some $\eta(t_0, r)$ $\forall r$

Then $dt'^2 = \eta^2\left(C^2 dt^2 - 2CE\, dt\, dr + E^2 dr^2\right]$

so $-\frac{dt'^2}{\eta^2}\frac{1}{C} = -C\,dt^2 + 2E\,dt\,dr - E^2/C\, dr^2$

$$\Rightarrow \quad -C\,dt^2 + 2E\,dt\,dr = -\frac{dt'^2}{\eta^2 C} + E^2/C\, dr^2$$

Substitute into ⊛ $\Rightarrow$

$$ds^2 = -\frac{dt'^2}{\eta^2 C} + \left(D + \frac{E^2}{C}\right)dr^2 + r^2 d\Omega^2$$

Let $B(t,r) \equiv \frac{1}{\eta^2(t,r)\,C(t,r)}$

$A(t,r) = D(t,r) + \frac{E^2(t,r)}{C(t,r)}$

& relabel t' by t $\Rightarrow$

$$\boxed{ds^2 = -B(t,r)\,dt^2 + A(t,r)\,dr^2 + r^2 d\Omega^2}$$