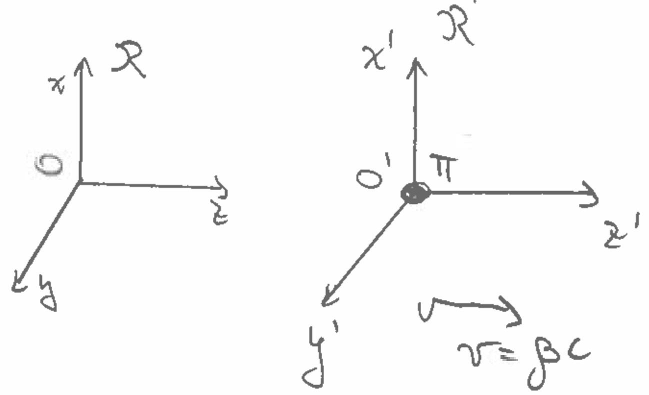
## I - Time dilatation - UR case

$L_0 = 3000\,m$

$\tau_0 = 2{,}6 \times 10^{-8}\,s$

$\gamma = 1000$

a) $ct = \gamma ct' + \beta\gamma z'$




$z' = 0$

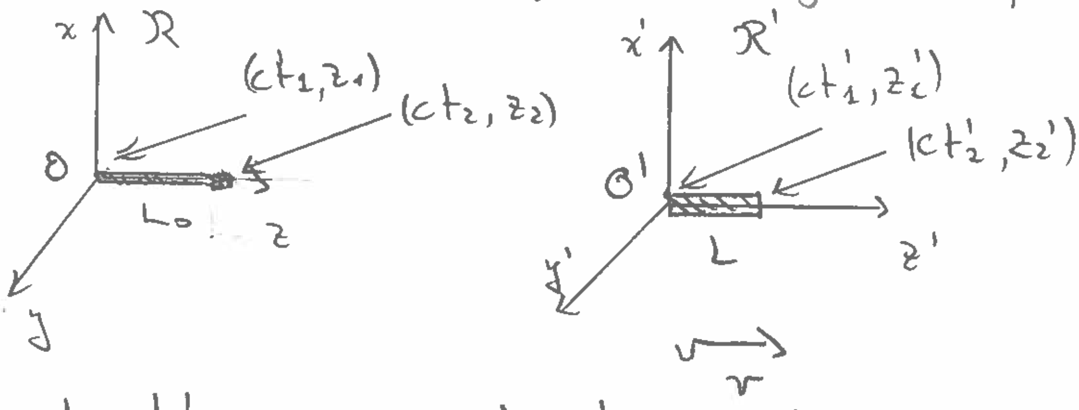
$\boxed{t = \gamma t'}$

(time dilution)

For the π mesons:

$\tau = \gamma \tau_0 = 2.6 \times 10^{-8} \times 10^{3} = 26\,\mu s$

b) A length is measured <u>simultaneously</u> on both sides of the object in a given reference.




At $t = t' = 0$ O and O' coïncide.

$z_1 = 0 \qquad z_2 = L_0 \quad ; \quad z_1' = 0 \qquad z_2' = L$

$t_1 = 0 \qquad t_2 = ? \quad ; \quad \underbrace{t_1' = 0 \qquad t_2' = 0}$

(crucial point: simultaneous events!)

$z_2 = \gamma(z_2' + \beta c t_2')$

$\Downarrow$

$L_0 = \gamma L$

$\boxed{L = \frac{1}{\gamma} L_0}$ (Length contraction)

Remark: It can be easily shown with a Lorentz transformation that

$$t_2 = \frac{\beta}{c} L_0 .$$

The two events are not (and cannot be!) simultaneous in $\mathcal{R}$.

c) $\gamma = 1000$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \iff \beta = \sqrt{1 - 1/\gamma^2}$$

$$\beta = \sqrt{1-10^{-6}} \simeq 1 - 0.5 \times 10^{-6} = 0.9999995$$

$v \simeq c$ (the UR approximation...)

$$t = \frac{L_0}{v} \simeq \frac{L_0}{c} = \frac{3 \times 10^3}{3 \times 10^8} = 1 \times 10^{-5} = 10 \mu s$$

d) $$\frac{\bar{N}(t)}{N_0} = e^{-t/\tau} = e^{\frac{10}{26}} = 0.68 = 68\%$$

68% of the $\pi$-mesons reach the detector.

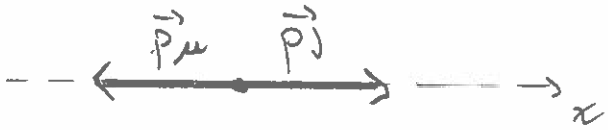
e) $$\frac{t}{\tau} = \frac{\gamma t'}{\gamma \tau_0} = \frac{t'}{\tau_0}$$

$\Downarrow$

$\frac{\bar{N}(t)}{N_0}$ does not change between references (fortunately...)

## II Decay at rest

a) $\pi^+ \to \mu^+ \nu_\mu$




$m_\nu = 0 \Rightarrow E_\nu = |\vec{p}_\nu| \equiv p^*$

Momentum conservation (in the CM):

$\vec{p}_\mu + \vec{p}_\nu = 0 \Rightarrow |\vec{p}_\mu| = p^*$

$\pi^+$ at rest $\Rightarrow E_\pi = m_\pi$

<u>4-vectors</u>

$$\begin{cases} p_\pi = (m_\pi, 0, \underbrace{[0, 0]}) \\ p_\mu = (E_\mu^*, -p^*) \\ p_\nu = (p^*, p^*) \end{cases}$$

the $p_y$ and $p_z$ components can be omitted.

Energy-momentum conservation (written in terms of 4-vectors):

$$p_\pi = p_\mu + p_\nu$$

$$p_\nu^2 = (p_\pi - p_\mu)^2$$

$$p_\nu^2 = p_\pi^2 - 2 p_\pi p_\mu + p_\mu^2$$

$$0 = m_\pi^2 - 2 m_\pi E_\mu^* + m_\mu^2$$

$$\boxed{\begin{aligned} E_\mu^* &= \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} \\ p^* &= m_\pi - E_\mu^* = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi} \end{aligned}}$$

## Useful definition:

The mass balance of the reaction

$$Q = \sum_i m_i - \sum_f m_f$$

(Mass difference between the initial- and final-state particles)

Kinetic energy

$$T_\mu^* = E_\mu^* - m_\mu = \frac{m_\pi^2 + m_\mu^2 - 2m_\pi m_\mu}{2m_\pi} = \frac{(\overbrace{m_\pi - m_\mu}^{Q})^2}{2m_\pi} = \frac{Q(m_\pi - m_\mu)}{2m_\pi}$$

$$T_\nu^* = p^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = \frac{(m_\pi - m_\mu)(m_\pi + m_\mu)}{2m_\pi} = \frac{Q(m_\pi + m_\mu)}{2m_\pi}$$

## Remarks:

1) $T_\mu^* + T_\nu^* = Q$ , as expected
2) The less massive particle takes more kinetic energy (as in classical mecanics ...)

## Numerical computation:

$E_\mu^* = 109.78$ MeV

$T_\mu^* = 4.12$ MeV

$T_\nu^* = E_\nu^* = p^* = 29.79$ MeV

b) Using the same expressions, replacing

$\mu \to {}^{12}C$ , $\pi \to {}^{12}C^*$ $(\nu \to \gamma)$

$m_\pi \to m_{c^*} = m_c + E_{ex}$

$m_\mu \to m_c$

$Q = E_{ex}$

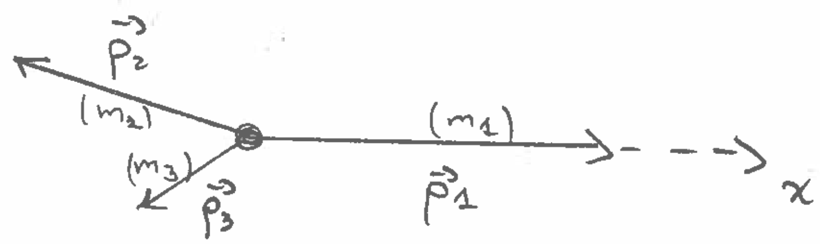
We obtain

$T_\alpha^* = 0.00088$ MeV

$T_\gamma^* = 4.43212$ MeV (almost all the kinetic energy)

c) This is a 3-body decay, and the full expressions may be long and complicated. Defining a strategy is therefore particularly important.

4-vectors (in the ref where the initial particle is at rest)




Initial state: $P_0 = (m_0, 0)$

Final state: $P_1 = (E_1, \vec{p}_1)$

$P_2 = (E_2, \vec{p}_2)$

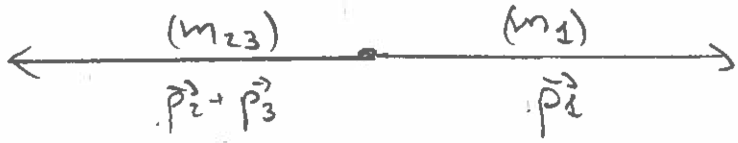
$P_3 = (E_3, \vec{p}_3)$

We will look of the maximum energy of particle "1".

Defining:

$P_{23} = P_2 + P_3$

$P_{23}^2 = (P_2 + P_3)^2 = m_{23}^2$,




we can consider particles "2" and "3" as a single object recoiling against "1"

$$P_0 = P_1 + P_{23}$$

$$P_{23}^2 = (P_0 - P_1)^2$$

$$m_{23}^2 = m_0^2 + m_1^2 - 2\underbrace{P_0 \cdot P_1}_{m_0 E_1}$$

$$E_1 = \frac{m_0^2 + m_1^2 - m_{23}^2}{2m_0}$$

For $E_1$ to be maximal, $m_{23}$ has to be minimal.
Clearly: $m_{23}^{min} = m_2 + m_3$.
(Think of it at the frame where the system "23" is at rest. To have $m_{23}^{min}$, "2" and "3" must be at rest in this frame)

$$\boxed{E_1^{max} = \frac{m_0^2 + m_1^2 - (m_2+m_3)^2}{2m_0}}$$

$$T_1^{max} = E_1^{max} - m_1 = \frac{(m_0 - m_1)^2 - (m_2+m_3)^2}{2m_0} =$$

$$= \frac{\overbrace{(m_0 - m_1 - m_2 - m_3)}^{Q}(m_0 - m_1 + m_2 + m_3)}{2m_0}$$

$$\boxed{T_1^{max} = \frac{Q\,(m_0 - m_1 + m_2 + m_3)}{2m_0}}$$

Another way to think of it: for $E_1$ to be maximal, $\vec{p}_1$ has to be maximal, and thus particles "2" and "3" must have the same direction (opposite to that of "1"). In this configuration, the maximal momentum of the system (or minimum $m_{23}$) is obtained when "2" and "3" move together, at the same velocity, as if they were a single particle of mass $m_{23} = m_2 + m_3$.

Using $Q = Q_\beta = m_n - m_p - m_e = 0.789$ MeV and attributing the role of "1" to each of the final state particles, we obtain

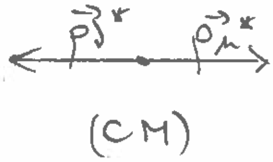
$T_p^{max} = 0.000752$ MeV $= 752$ eV (tiny!)

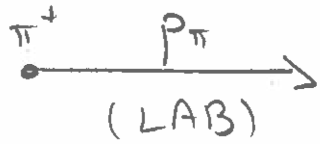
$T_e^{max} \simeq T_\nu^{max} = 0.7822 \approx Q$

## III - Decay in flight

Using the results of Ex. II a, and boosting them to the $\pi^+$ frame, the maximum momentum of the $\mu^+$ is obtained when its flight direction is the same as that of the $\pi^+$. The minimum is obtained for the opposite direction

$E_\mu^{max}$ :




$E_\mu^{min}$ :


$(|\vec{p}_\mu^{\,*}| = |\vec{p}_\nu^{\,*}| \equiv p^*)$

$$E_i^{LAB} = \gamma\left(E_i^{CM} + \beta p_i^{CM}\right)$$

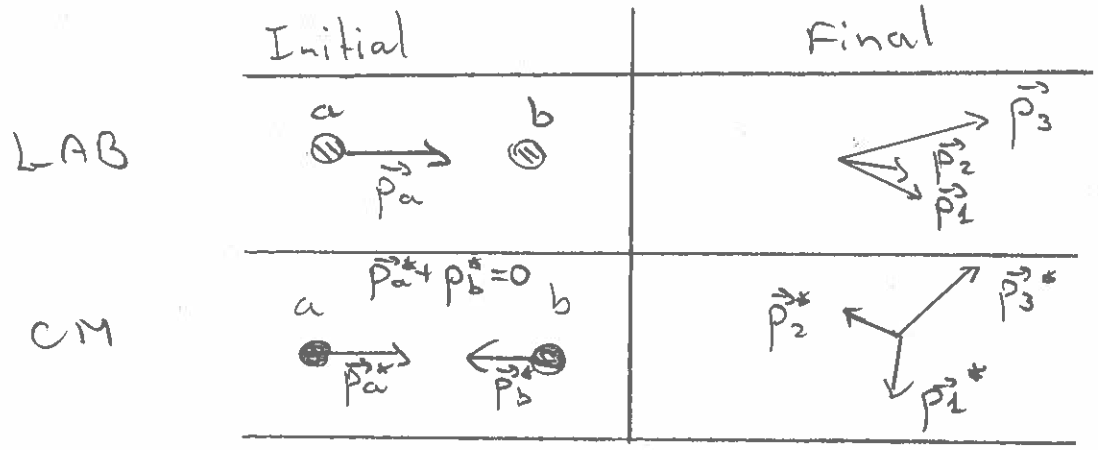
$$E_\mu^{max} = \gamma\left(E_\mu^* + \beta p^*\right) = \gamma\left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} + \beta\,\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right) =$$

$$= \frac{\gamma}{2m_\pi}\left(m_\pi^2(1+\beta) + m_\mu^2(1-\beta)\right)$$

$$E_\mu^{min} = \gamma\left(E_\mu^* - \beta p^*\right) = \frac{\gamma}{2m_\pi}\left(m_\pi^2(1-\beta) + m_\mu^2(1+\beta)\right)$$

# IV Threshold of particle production

We first look at the problem in general, then apply the results on the particular cases




Initial LAB
$$\begin{cases} P_a = (E_a, p_a) \\ P_b = (m_b, 0) \end{cases}$$

Final CM at threshold
$$\begin{cases} P_1^* = (m_1, 0) \\ P_2^* = (m_2, 0) \\ P_3^* = (m_3, 0) \\ \vdots \end{cases}$$

At threshold, all the final state particles are produced <u>at rest</u> <u>in the CM</u>.
$(\vec{p}_1^{\,*} = \vec{p}_2^{\,*} = \vec{p}_3^{\,*} = \vec{0})$

Knowing that the norm is invariant:

$$(P_a + P_b)^2 = \left(\sum_f P_f\right)^2$$

$$(E_a + E_b)^2 - p_a^2 = \left(\sum_f m_f\right)^2$$

$$m_a^2 + m_b^2 - 2E_a m_b = \left(\sum_f m_f\right)^2$$

$$E_a = \frac{\left(\sum_f m_f\right)^2 - (m_a + m_b)^2}{2m_b}$$

$$E_a = T_a^{th} + m_a$$

$$T_a^{th} = \frac{\left(\sum_f m_f\right)^2 - \left(\sum_i m_i\right)^2}{2 m_b} = \frac{\overbrace{\left(\sum_f m_f + \sum_i m_i\right)}^{\equiv \sum m} \cdot \overbrace{\left(\sum_f m_f - \sum_i m_i\right)}^{= -Q}}{2 m_b}$$

$$\boxed{T_a^{th} = \frac{-Q \cdot \sum m}{2 m_b}}$$

(This is not a formula to learn by heart. You should know how to find it.)

a) ⊛ $pp \to pp\pi^0$

$$-Q = m_{\pi^0} \simeq 135 \text{ MeV}$$

$$\sum m = 4 m_p + m_{\pi^0} \simeq 4 \times 938.3 + 135.0 = 3888.2 \text{ MeV}$$

$$T_p^{th} = \frac{135 \times 3888.2}{2 \times 938.3} = 279.7 \text{ MeV}$$

⊛ $pp \to pn\pi^+$

$$T_p^{th} \simeq 292{,}4 \text{ MeV}$$

b) $-Q = 2 m_p \quad ; \quad \sum m = 6 m_p$

$$T_p^{th} = \frac{\cancel{2 m_p} \times 6 m_p}{\cancel{2 m_p}} = 6 m_p$$

c) 1. $-Q = 2 m_W - 2 m_e \simeq 2 m_W$

$$\sum m \simeq 2 m_W$$

$$T_{e^+}^{th} \simeq \frac{2 \cancel{4} m_W^2}{8 m_e} = \frac{2 m_W^2}{m_e} \simeq 2 \frac{(80.1 \times 10^9)^2}{0.5 \times 10^6} \simeq 2.6 \times 10^{17} \text{ eV}$$

2. $E_e = m_W$

$$T_e = m_W - m_e \simeq m_W = 80.1 \text{ GeV}$$

The fixed target scenario is totally irrealistic to obtain in a collider ($\sqrt{s} = 13 \times 10^{12}$ eV in the LHC). The second scenario was realized at LEP.

# $\gamma$- Photoproduction of $\pi$ mesons

a) $T_\gamma^{th} = p_\gamma^{th} \simeq 151.5$ MeV

b) $E_\gamma = k_B T = 2.6 \times 10^{-4}$ eV $= 0.26 \times 10^{-9}$ MeV

$$P_\gamma = (E_\gamma, -E_\gamma)$$
$$P_p = (E_p, p_p)$$
$$(P_\gamma + P_p)^2 = (m_n + m_\pi)^2$$
$$m_p^2 + 2(E_p E_\gamma + p_p E_\gamma) = (m_n + m_\pi)^2$$
$$m_p^2 + 2E_\gamma(E_p + p_p) \simeq (m_n + m_\pi)^2$$
$$E_p + p_p = \frac{(m_n + m_\pi)^2 - m_p^2}{2E_\gamma} \simeq 0.55 \times 10^{15} \text{ MeV}$$

The proton is UR, as $E_p + p_p \gg m_p$

$$E_p \simeq p_p \simeq 0.27 \times 10^{15} \text{ MeV}$$

Protons with energy above this $E_p$ will react with the CMBR. Thus they can not travel long distances in the universe, which is totally opaque for them. What we just computed is related to the GZK (Greisen - Zatsepin - Kuzmin) limit in cosmology.

c) $P_\gamma = (p, p)$ ("projectile" $\gamma$)

$P_{\gamma 3} = (E_\gamma, -E_\gamma)$ ($\gamma$ from CMBR)

$$(p + E_\gamma)^2 - (p - E_\gamma)^2 = 4m_e^2$$
$$\cancel{4}\, p E_\gamma = \cancel{4} m_e^2$$
$$p = \frac{m_e^2}{E_\gamma} = 1.01 \times 10^9 \text{ MeV}$$

# VI The decay $\pi^0 \to \gamma\gamma$

a) In the LAB frame:

$$P_\pi = (E_\pi, \vec{p}_\pi)$$
$$P_{\gamma 1} = (p_1, \vec{p}_1)$$
$$P_{\gamma 2} = (p_2, \vec{p}_2)$$

$$(P_{\gamma 1} + P_{\gamma 2})^2 = P_\pi^2$$

$$\cancel{2m_\gamma^2}^{\to 0} + 2 P_{\gamma 1} P_{\gamma 2} = m_\pi^2$$

$$2 ( p_1 p_2 - \underbrace{\vec{p}_1 \cdot \vec{p}_2}_{= p_1 p_2 \cos\theta} ) = m_\pi^2$$

$$2 p_1 p_2 (1 - \cos\theta) = m_\pi^2$$

$$2(1-\cos\theta) = 4 \sin^2 \frac{\theta}{2}$$

$$\sin^2 \frac{\theta}{2} = \frac{m_\pi^2}{4 p_1 p_2}$$

$$\boxed{\theta = 2 \arcsin \frac{m_\pi}{2\sqrt{p_1 p_2}}}$$

$\theta$ is minimum when $f \equiv p_1 p_2$ is maximum

$$p_2 = E_\pi - p_1$$
$$f(p_1) = p_1 (E_\pi - p_1)$$
$$f'(p_1) = E_\pi - 2p_1 = 0$$

$\boxed{p_1 = p_2 = \frac{E_\pi}{2}}$ (at the minimum of $\theta$)

b) ⊛ $\pi^0$ at rest:

$E_\pi = m_\pi$

$p_1 = p_2 = \frac{m_\pi}{2}$

$\Theta = 2 \arcsin 1 = 2 \times \frac{\pi}{2} = \pi$

As it should be... ($\gamma_1, \gamma_2$ back to back in the $\pi^0$ rest frame)

⊛ If the $\pi^0$ is not at rest

$E_\pi = \gamma m_\pi$ ; $p_1 = p_2 = \frac{\gamma m_\pi}{2}$

$\Theta = 2 \arcsin \frac{\cancel{m_\pi}}{\gamma \cancel{m_\pi}} = 2 \arcsin \frac{1}{\gamma}$

$\rightarrow$ UR $\pi^0$ : $\gamma \gg 1$

$\Theta \simeq \frac{2}{\gamma}$

$\rightarrow$ $T(\pi^0) = m_\pi \Rightarrow E_\pi = 2 m_\pi \Rightarrow \gamma = 2$

$\Theta = 2 \arcsin \frac{1}{2} = \frac{\pi}{3}$

## VII Two-body decay

a) Results:

$$E^*_{J/\psi} = \frac{m_B^2 + m_{J/\psi}^2 - m_K^2}{2 m_B} \simeq 3.53 \text{ GeV}$$

$$T^*_{J/\psi} = E_{J/\psi} - m_{J/\psi} \simeq 0.43 \text{ GeV}$$

$$E^*_K \simeq 1.77 \text{ GeV}$$

$$T^*_K \simeq 1.26 \text{ GeV}$$

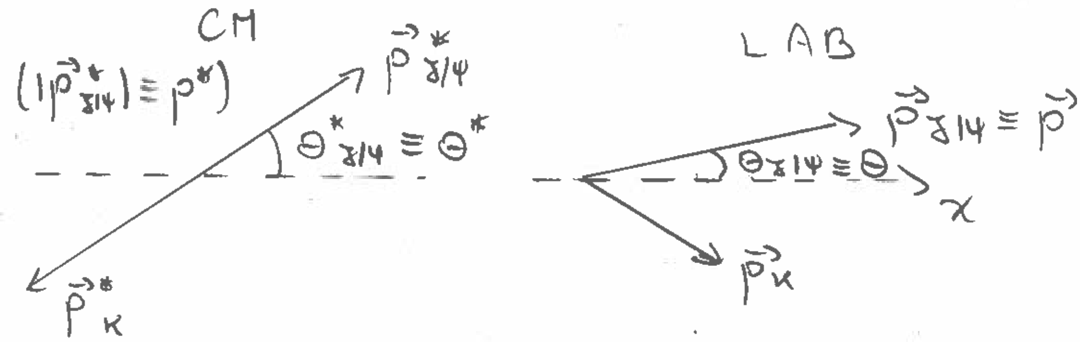
b) $E_B = 45$ GeV

$$\gamma = \frac{E_B}{m_B} \simeq 8.5$$

1) $$E_{J/\psi}^{\substack{min \\ max}} = \gamma\left(E^*_{J/\psi} \mp \beta p^*\right)$$

$p^* = \sqrt{E^{*2}_{J/\psi} - m^2_{J/\psi}}$

2)




CM: $\underline{p}^*_{J/\psi} = (E^*_{J/\psi}, \; p^* \cos\theta^*, \; p^* \sin\theta^*, \; 0)$

LAB: $\underline{p}_{J/\psi} = (E_{J/\psi}, \; p\cos\theta, \; p \sin\theta, \; 0)$

$$\begin{cases} p\cos\theta = \gamma\left(p^*\cos\theta^* + \beta E^*_{J/\psi}\right) \\ p\sin\theta = p^*\sin\theta \end{cases}$$

$\Downarrow$

$$\boxed{\tan\theta = \frac{p^*\sin\theta^*}{\gamma p^*\cos\theta^* + \beta\gamma E^*_{J/\psi}} = \frac{\sin\theta^*}{\gamma\cos\theta^* + \beta\gamma \frac{E^*_{J/\psi}}{p^*}}}$$

$(\beta\gamma = \sqrt{\gamma^2 - 1} \simeq 8.44)$