Correction of exercise sheet № 3 - Collisions and decays

1 Correction of the Cockroft-Walton exercise

1. $\frac{|p_A|}{M_A} = \ll 1$, then the total energy can be written as:

$$E_A = \sqrt{m_A^2 + p_A^2} = m_A\sqrt{1 + \frac{p_A^2}{m_A^2}} \approx m_A + \frac{p_A^2}{2\,m_A} \equiv m_A + K_A,$$

   where K_A is the kinetic energy. Generally speaking the K is given by: $K = E - m = (\gamma - 1)m$.

2. The 4-vectors are in the lab

$$\begin{aligned} q_{Li} &= (M_{Li}, 0, 0, 0) \\ q_p &= (E_p \approx M_p + \frac{p_z^2}{2\,m}, 0, 0, p_z) \end{aligned} \tag{1}$$

   Total mass of the system:

$$\begin{aligned} (M_*)^2 &= (q_{Li} + q_p)^2 = (m_{Li} + E_p)^2 - pz^2 \\ &= m_{Li}^2 + E_p^2 - p_z^2 + 2\,m_{Li}E_p = m_{Li}^2 + m_p^2 + 2\,m_{Li}\,(m_p + K_p) \\ &= (m_{Li} + m_p)^2 + 2\,m_{Li}\,K_p \approx (m_{Li} + m_p)^2 \end{aligned} \tag{2}$$

3. The boost into the colliding particle rest frame is along the z axis and is given by:

$$\gamma_* = \frac{E_*}{M_*} \approx \frac{m_{Li} + E_p}{m_{Li} + m_p} \tag{3}$$

   Similarly $\gamma_* \times \beta_*$ is given by:

$$\gamma_* \times \beta_* = \frac{p_*}{M_*} \approx \frac{p_z}{m_{Li} + m_p} \tag{4}$$

4. The momentum of the proton in the rest frame is given by

$$\begin{aligned} E_{p*} &= \gamma_* E_p - \gamma_*\beta_* p_z = \frac{E_*}{M_*}E_p - \frac{p_*}{M_*}p_z \\ &= \frac{m_{Li} + E_p}{M_*}E_p - \frac{p_z}{M^*}p_z \\ &= \frac{m_{Li}E_p + m_p^2}{M_*} \\ K_* + m_p &= \frac{m_{Li}(m_p + K) + m_p^2}{m_{Li} + m_p} \\ K_* &= \frac{m_{Li}}{m_{Li} + m_p}K < K!!!! \end{aligned} \tag{5}$$

   Note that if the target is a proton: $K_* = K/2$, K_* being the effective energy available in the center of mass frame for the reaction, we need to provide 2 times more energy in the lab. The best is to have make the lab and the rest-frame similar by colliding beam-beam. We could have neglected from the beginning the kinetic energy of the proton from the beginning and assume the total system is at rest in the lab .

5. In the rest-frame $*$, 4-vector conservation gives (noting $_i$ the quadri momentum of α_i):

$$\begin{array}{rcccl} E_* & = & M_* & = & E_1 + E_2 \\ \vec{k}_* & = & \vec{0} & = & \vec{p}_1 + \vec{p}_2 \end{array} \tag{6}$$

The 2 alpha particles are back-to-back, and we have: $|\vec{p}_1| = |\vec{p}_2| = p_*$, thus:

$$M_* = \sqrt{m_1^2 + p_*^2} + \sqrt{m_2^2 + p_*^2}$$

We get to:

$$\begin{aligned} \sqrt{m_2^2 + p_*^2}^{\,2} &= \left(M_* - \sqrt{m_1^2 + p_*^2}\right)^2 \\ m_2^2 + p_*^2 &= M_*^2 + m_1^2 + p_*^2 \; - \; 2 * \sqrt{m_1^2 + p_*^2}\, M_* \\ \sqrt{m_1^2 + p_*^2}^{\,2} &= (M_*^2 + m_1^2 - m_2^2)^2/(4\, M_*^2) \\ p_* &= \frac{\sqrt{M_*^4 + m_1^4 + m_2^4 - 2\, m_1^2 m_2^2 - 2\, M_*\, m_1^2 - 2\, M_*^2 m_2^2}}{2\, M_*} \end{aligned} \tag{7}$$

6. In the case of a decay to α particles, $m_1 = m_2 \equiv m_\alpha$

$$p_* = \frac{\sqrt{M_*^2 - 4\, m_\alpha^2}}{2}$$

$$\begin{aligned} K_*[\alpha] &= \frac{p_*^2}{2\, M_\alpha} \\ &= \frac{(M_{Li} + M_p)^2 - 4\, M_\alpha^2}{8\, M_\alpha} \\ &= 8.46 \text{ MeV} \end{aligned}$$

7. A much simpler resolution can be obtained by using the energy conservation in the center of mass:

$$E_*[\alpha_1] + E_*[\alpha_2] = 2 \times E_*[\alpha] = M_*$$

and then:

$$\begin{aligned} K_*[\alpha] &= E_*[\alpha] - M_\alpha \\ &= \frac{(M_{Li} + M_p) - 2\, M_\alpha}{2} \\ &= 8.45 \text{ MeV} \end{aligned}$$

2 Correction of the luminosity exercise

1. The instantaneous luminosity is given by the formula in the course.

$$\mathcal{L} = \frac{n_{bunch}\ f\ N_1\ N_2}{4\sqrt{\epsilon_x^* \epsilon_y^* \beta_x^* \beta_y^*}} \times R_\phi \quad \text{with} \quad R_\phi = \frac{1}{\sqrt{1 + \phi^2}},$$

with N_1 and N_2 the number of particles per bunch in beam 1 and beam 2. In this case, both are proton beams with $N_p = 1.1 \times 10^{11}$ proton per bunch, f is the frequency of collision, the spacing between bunches being 25 ns, the collision frequency is therefore $f = 1/(25\ \mathrm{ns}) = 40$ MHz. Putting it all together.

$$\begin{aligned}
\mathcal{L} &= \frac{2808 \times (1.1)^2 \times 10^{22} \times 40 \times 10^6}{4\sqrt{3.75 \times 10^{-4}}^2 \times 55^2} \times \frac{1}{\sqrt{1+0.64^2}}\ \mathrm{cm}^{-2}.\mathrm{s}^{-1} && (8)\\
\mathcal{L} &= 13.6 \times 10^{3+22+1+6}/(8.25 \times 10^{-2}) \times 0.842\mathrm{cm}^{-2}.\mathrm{s}^{-1} && (9)\\
\mathcal{L} &\approx 1.4\ 10^{34}\ \mathrm{cm}^{-2}.\mathrm{s}^{-1} && (10)
\end{aligned}$$

2. The barn is the appropriate unit for cross section measurement 1 b = 10^{-24} cm^2. Therefore a nanobarn is 1 nb = 10^{-33} cm^2, which gives:

$$\mathcal{L} \approx 14\ \mathrm{nb}^{-1}.\mathrm{s}^{-1} \tag{11}$$

3. This gives the total integrated luminosity just by multiplying with a year duration.

$$\mathcal{L} = 14\ \mathrm{nb}^{-1}.\mathrm{s}^{-1} \times 10^7\mathrm{s} = 140\ \mathrm{fb}^{-1} \tag{12}$$

Note that the actual integrated luminosity recorded by the CMS experiment was in 2016 36 fb^{-1}, 45 fb^{-1} in 2017 and 60 fb^{-1} in 2018.

4. The total number of Higgs bosons produced per year in a single experiment is therefore:

$$N_H = \sigma(pp \to H) \times L = 50000\mathrm{fb} \times 140\mathrm{fb}^{-1} = 7 \times 10^6 \tag{13}$$

5. The total number of Higgs boson produced per year which are potentially usable in the diphoton channel is thus:

$$N_{H\gamma\gamma} = \sigma(pp \to H) \times L\mathcal{B}(H \to \gamma\gamma) \times \epsilon \approx 7.10^6 \times 2.10^{-3} \times 0.5 = 7000 \tag{14}$$