M2 NPAC – Particle physics (2023-2024)

Exercise sheet № 4 - Introduction to strong interaction: isospin, SU(3)

As usual, it is necessary to use the PDG to solve the exercises. Here, in particular, it would be useful to have in mind some properties of the light hadrons (containing u, d and s quarks), and especially strange hadrons (e.g. Λ, $\Sigma^{\pm,0}$, $\Xi^{-,0}$, $K^{\pm}$, K^0, $\overline{K}^0$). It is therefore recommended to go through the corresponding tables before starting.

Exercise 1

Why is the reaction $\Lambda \to K^0 X$ not observed? In other words, show that such a particle X cannot exist.

Exercise 2: the deuteron

The deuteron is a bound state of a proton (p) and a neutron (n). This is the only observed bound state of two nucleons. It has only one energy level: the fundamental. Spectroscopic measurements show that its J^P is 1^+.

a) What are the quantum numbers s and ℓ of the deuteron?
b) What are the quantum numbers *s*, ℓ and J^P that would be possible for bound states of p-p and n-n, if these states existed?
c) Experiments of nucleon-nucleon scatterings and other experimental results show that strong interaction between nucleons does not depend on their electrical charge. How this observation is compatible with the non-existence of bound states n-n and p-p?
d) Why is 0-isospin attributed to the deuteron?

Exercise 3

An experiment of p-p and p-n scatterings at a given energy in the CM frame yielded these two measurements of total cross sections:

$$\sigma^+ = \sigma(pp \to d\pi^+) = (3.15 \pm 0.22)\ \text{mb}$$
$$\sigma^0 = \sigma(np \to d\pi^0) = (1.5 \pm 0.3)\ \text{mb}$$

Supposing that the reactions are due to a strong interaction process, show that isospin conservation allows predicting the ratio of these cross sections (NB. $I_d=0$, see exercise 2). Use the notion of isospin amplitude $T_I \equiv \langle I,I_3 | H_{int} | I,I_3 \rangle$, where H_{int} is the Hamiltonian of the interaction. Explain the fact that this amplitude does not depend on I_3.

Exercise 4

In this exercise we explore a few aspects of $\pi^{\pm}$ p scattering.

a) Which final states can be produced, containing only one pion and one nucleon (N)?

When the energy in the CM frame $\sqrt{s}$ is close to the Δ baryon mass, the cross section presents a peak; the process becomes resonant (see Figure 1). We consider that the Δ particle is produced as an intermediate state between the initial and the final states of the interaction process ("intermediate resonance"). For the properties of the Δ, see Figure 1 and the PDG Booklet.

b) Which interaction is responsible to the creation and the decay of the Δ resonance? What is the lifetime of the Δ?

c) Considering only isospin-related quantum numbers, write the states of the systems (π^+,p), (π^-,p), (π^0,n) in the basis of eigenvectors common to I^2 and I_3, where I and I_3 are, respectively, the total isospin and its projection.

d) We define:

$$\sigma^+ = \sigma(\pi^+ \, p \rightarrow \pi^+ \, p),$$
$$\sigma^0 = \sigma(\pi^- \, p \rightarrow \pi^0 \, n),$$
$$\sigma^- = \sigma(\pi^- \, p \rightarrow \pi^- \, p).$$

Using $\sigma = |\langle f | H_{int} | i \rangle|^2$, write these three total cross sections as functions of the isospin amplitudes $T_{I=1/2}(\pi \, N)$ and $T_{I=3/2}(\pi \, N)$ (where $T_I \equiv \langle I, I_3 | H_{int} | I, I_3 \rangle$).

e) In the case where these processes are resonant with $\sqrt{s}\sim 1.2$ GeV, explain why $T_{I=1/2}(\pi \, N)=0$. Deduce the value of the ratio

$$\sigma(\pi^+ \, p \rightarrow \Delta^{++} \rightarrow \pi \, N) \; / \; \sigma(\pi^- \, p \rightarrow \Delta^0 \rightarrow \pi \, N)$$

and compare your result to the distributions of Figure 1.

f) We also notice other resonances on Figure 1. What is *a priori* the isospin of the two resonances located between $\sqrt{s}$=1.4 GeV and $\sqrt{s}$=1.8 GeV?

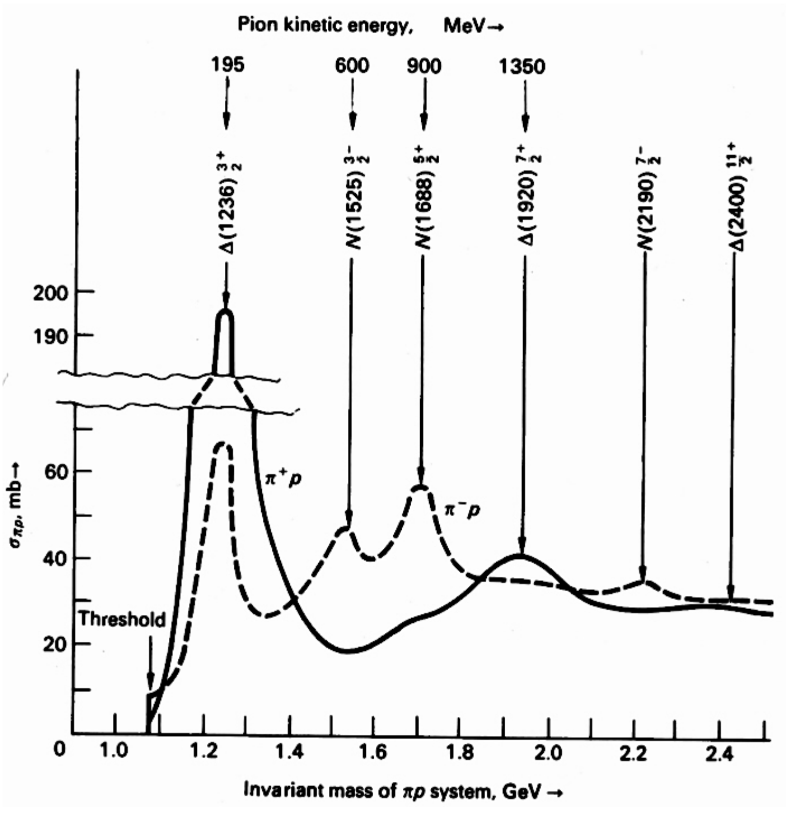


Figure 1: Cross section as a function of the invariant mass of πp system.

<u>Exercise 5</u> Strangeness model: forbidden-allowed

a) Is the reaction $\Sigma^0 \to \Lambda\gamma$ allowed? If it is, by which interaction?
b) Is it possible to observe the reaction $\Lambda \to \gamma X$? (in other words, is there a possible candidate for the X particle in such a reaction?)
c) Is the reaction $\pi^- p \to \Lambda\pi^0$ allowed? If it is, indicate by which interaction and make a qualitative comment on the corresponding cross section.
d) For all the allowed reactions in a), b) and c) above, draw the Feynman diagrams.

<u>Exercise 6</u>

In this exercise we consider the following strong-interaction decays of the ϕ meson: $\phi \to K^+K^-$ and $\phi \to \rho\pi$.

Estimate the ratio of the corresponding decay widths: $R=\Gamma(\phi \to \rho\pi)/\Gamma(\phi \to K^+K^-)$. Suppose that the matrix elements are the same for these two decays[1].

The experimentally measured value of R is 0.35±0.2. This result was at the origin of the theoretical puzzle that led Zweig to postulate the existence of the s-quark. Explain why.

<u>Exercise 7</u> The spin of the pion

The spin of the charged pion was determined by measuring the total cross sections of the reactions $pp \to \pi^+ d$ and $\pi^+ d \to pp$. At the time of this experiment, spins of the proton (1/2) and of the deuteron (1) were already known.

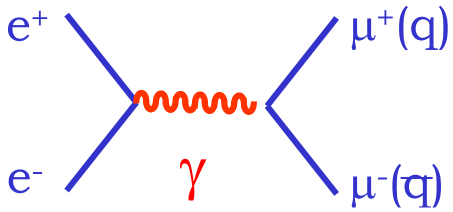
a) What is the transformation that allows passing from one reaction to the other? What would be the relation between the matrix elements of these two reactions if this transformation was a symmetry? How is the cross section affected when we take into account the non-distinguishability of the two protons in the final state of the second reaction?
b) For the first reaction, a proton beam with a kinetic energy of T_p=340 MeV was used with a fixed target of protons. For the second reaction, what should be the kinetic energy of a π^+ in a beam colliding with a fixed target of deuteron to ensure the same total energy in the centre of mass frame? Explain why is it necessary to have the same total energy in the CM frame in order to compare these two reactions.
c) Knowing that these reactions are allowed, show that the spin of the pion cannot be half integer.
d) Express the cross sections of these two reactions as functions of the spins of the proton and the deuteron. Using at the ratio of these cross sections together with the results:
   - $\sigma(pp \to \pi^+ d)$=0.18±0.06 mb,
   - $\sigma(\pi^+ d \to pp)$= 3.1±0.3mb,

   obtained for the same total energy in the CM frame (the energy calculated in b above), deduce the value of the spin of the pion. Compute the uncertainty on this value, considering that the given uncertainties are the dominant ones and neglecting the correlations between them.

<u>Exercise 8</u> The R-ratio

Consider the following processes of e^+e^- annihilation.

[1] We precise that our hypothesis is done on matrix elements of transition between given states of spin. These elements are supposed to be equal for all spin states.



We can show that:

$$\sigma_{\mu^+\mu^-} \equiv \sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3s},$$

$$\sigma_{q\bar{q}} \equiv \sigma\left(e^+e^- \to q\bar{q}\right) = N_c e_q^2 \sigma\left(e^+e^- \to \mu^+\mu^-\right).$$

The e_q factor represents the electric charge of quarks, and the factor N_c appears because there is one diagram per quark color (N_c =3).

a) Give the formula of the R-ratio, defined as $\sigma(e^+e^-\to\text{hadrons})/\sigma_{\mu\mu}$.

b) Figure 2 shows the experimentally measured R-ratio as a function of √s. There are two "steps" between the levels indicated by horizontal lines, in the region 1-20 GeV. What do these steps correspond to? Note that they are located near the sharp spikes corresponding the resonances Φ, J/ψ, ψ' and Υ (for their characteristics, see the PDG Booklet). Check that the three values in the figure agree with the theoretical expression of R, found above. Explain the importance of this experimental result.

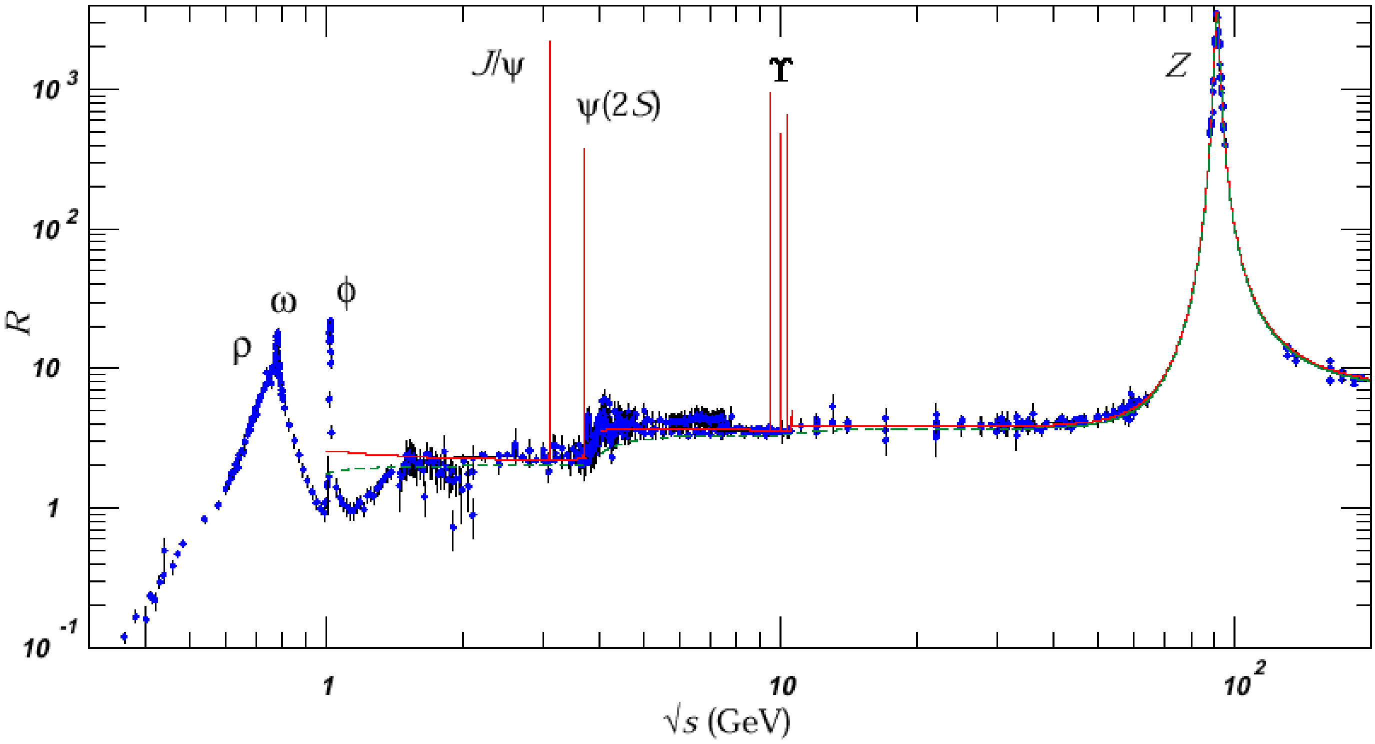


Figure 2 : The *R-ratio.*

<u>Exercice 9</u>

We consider four decay processes:

(i) $\omega \to \eta\, \pi^0$

(ii) $\rho^0 \to \eta\, \pi^0$
(iii) $\rho^+ \to \eta\, \pi^+$
(iv) $\rho^+ \to \pi^0 \pi^+$

a) Show that the decays (i) and (ii) cannot occur by strong or electromagnetic interactions.
b) Taking into account the relevant conservation laws except for G-parity, by which interaction could proceed the decays (iii) and (iv)?
c) Supposing that the matrix elements in the processes (iii) and (iv) are the same, estimate the ratio of branching fractions of these two decays. Compare your result with the information given in the PDG concerning the partial widths of the two decays. Is the same-matrix-element hypothesis acceptable? Make any comment that seems useful.
d) Compute the G-parity eigenvalues for the particles and states involved in processes (iii) and (iv).
e) Answer again question b), this time using G-parity. Comment your observations from question c).
f) Draw the Feynman diagrams of processes (i), (iii) and (iv).
g) Explain why the G-parity is conserved in strong-interaction processes, unlike in weak- or electromagnetic-interaction processes.

<u>Exercice 10</u> antiquarks in SU(3) and SU(2)

Assuming an SU(3) symmetry for the three light quarks (u,d,s), we saw that the symmetry transformation is $\left|q_i'\right\rangle = \sum_{j=1}^{3} U_{ij}\left|q_j\right\rangle$, where $\left|q_j\right\rangle$ represent the $\left|u\right\rangle$, $\left|d\right\rangle$ and $\left|s\right\rangle$ states for j=1, 2 and 3, respectively. We recall that U_{ij} are the elements of the general 3x3 transformation matrix of SU(3), written as $U = e^{i\vec{\theta}\cdot\vec{T}}$, where $\vec{T}$ is the vector of SU(3) generators, and $\vec{\theta}$ is the vector of the corresponding 8 real coefficients. From quantum field theory, we know that the antiparticle states must transform via complex conjugate matrices. In our case, for antiquarks: $\left|\bar{q}_i'\right\rangle = \sum_{j=1}^{3} U_{ij}^*\left|\bar{q}_j\right\rangle$. In this exercise we use the last expression to show some features of antiquarks.

a) Show that the SU(3) generators for antiquarks are the matrices $-\vec{T}^*$.
b) Representing the antiquark states by the column vectors

$$\left|\bar{u}\right\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad , \quad \left|\bar{d}\right\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad , \quad \left|\bar{s}\right\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad ,$$

deduce that they are represented in the (I_3, Y) plane by the SU(3) multiplet $\bar{3}$.

The SU(2) subgroup of SU(3) (generated by T_1, T_2 and T_3) has a special property. It turns out that there exists a unitary transformation, noted R_I, that changes $-T^*_i \to T_i$ (i=1,2,3). We recall that to accomplish this, R_I must satisfy

$$R_I(-T_i^*)R_I^\dagger = T_i \quad , \quad \text{for } i = 1,2,3.$$

c) Show that the R_I matrix must commute with T_2 and anti-commute with T_1 and T_3. Show that the matrix is

$$R_I = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From now, to simplify, we can work in the 2x2 sub-space corresponding to SU(2).

d) Show that, inside the SU(2) subgroup, instead of applying the generators $-\vec{T}^*$ on antiquarks, one can simply use T_1, T_2 and T_3 on both quarks and antiquarks, provided that the antiquark states are replaced by $R_I\left|\bar{q}\right\rangle$.

e) Write explicitly $R_I\left|\bar{q}\right\rangle$ for $\left|\bar{u}\right\rangle$ and $\left|\bar{d}\right\rangle$. This is the $\bar{2}$ multiplet of SU(2).

f) Using these states, extract I_3 for these two antiquark states and check that they correspond to the values found in b).

The results of e) and f) allow us to change known isospin states involving particles only into states involving particles and antiparticles.

g) Write the isospin states that would be observed for the (non-physical) mesons made of the two light quarks.

h) From these, get the corresponding states made of one light quark and one light antiquark. Compare with the expression of the isospin triplet, and with the quark contents of the π mesons and the η.