Angular distributions and helicity formalism

The helicity formalism is used to obtain angular distributions of final-state particles in processes of collision (interaction) and decay.

Here, we aim to obtain the term $|d^J_{\lambda i\,\lambda f}(\theta)|^2$ in the expressions of σ et Γ.

We will use as an illustrative example the simple case $\Lambda \to p\pi$.

Helicity: reminder

The helicity is defined as $\hat{H} = \vec{n} \cdot \hat{\vec{S}}$, with $\vec{n} = \dfrac{\vec{p}}{|\vec{p}|}$.

It is important to note that, if $\vec{L}$ is the orbital angular momentum and $\vec{J} = \vec{S} + \vec{L}$, we have

$$\hat{H} = \vec{n} \cdot \vec{S} = \vec{n} \cdot \vec{J}, \quad \text{because} \quad \vec{p} \cdot \vec{L} = \vec{p} \cdot \left(\vec{r} \wedge \vec{p}\right) = 0.$$

The eigenvalues of H are $\lambda \mathrm{h} = -s\mathrm{h}, \ldots\ldots, (s-1)\mathrm{h}, s\mathrm{h}.$

Given that H is a scalar (dot) product, the helicity is invariant under rotation.

Rotation of an angular-momentum state – the $d^j_{m,\,m'}(\theta)$ functions

We take the quantification-axis z in the plane of the page, and the perpendicular axis, pointing upwards, as y. A rotation by an angle θ about the y-axis is obtained by the operator

$$R = e^{-i\theta J_y}.$$

An angular-momentum eigenstate, $|\, j, m >$, transformed by such a rotation is a linear combination of states $|\, j, m' >$, with $m' = -j, -j+1, \ldots , j$. The coefficients of this linear combination depend on the angle θ and on the quantum numbers j, m and m'. We denote these coefficients as $d^j_{m,\,m'}(\theta)$, and thus:

$$e^{-i\theta J_y}\left|j,m\right\rangle = \sum_{m'} d^j_{m,m'}\left(\theta\right)\left|j,m'\right\rangle.$$

The quantum number j is unchanged, as $[R, J^2]=0$. The projection of this state on $< j, m' |$ is:

$$\left\langle j,m'\right| e^{-i\theta J_y}\left|j,m\right\rangle = d^j_{m,m'}\left(\theta\right).$$

The functions $d^j_{m,m'}(\theta)$ are therefore simply interpreted as the elements of the rotation matrix R.

Below we will explicitly obtain these functions for $j=½$. In this case:

$$J_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$e^{-i\theta J_y} = \mathbf{1} - i\theta J_y - \frac{\theta^2}{2!}J_y^2 + i\frac{\theta^3}{3!}J_y^3 + \cdots = \mathbf{1} - i\frac{\theta}{2}\sigma_y - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2\sigma_y^2 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3\sigma_y^3 + \cdots$$

Using the fact that $\sigma_y^2 = 1$, we can write:

$$e^{-i\theta J_y} = \mathbf{1}\cdot\cos(\theta/2) - i\sigma_y \sin(\theta/2) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Denoting:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ; \quad \left\langle\frac{1}{2},\frac{1}{2}\right| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad ; \quad \left\langle\frac{1}{2},-\frac{1}{2}\right| = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

we finally obtain:

$$d^{1/2}_{1/2,1/2}(\theta) = d^{1/2}_{-1/2,-1/2}(\theta) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\theta/2),$$

$$d^{1/2}_{-1/2,1/2}(\theta) = -d^{1/2}_{1/2,-1/2}(\theta) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sin(\theta/2).$$

The $d^{j}_{m,m'}(\theta)$ functions for higher angular momenta are technically more difficult to compute, but are obtained in a similar way. These functions are detailed, for different values of j, in the Clebsch-Gordan coefficients table.

As we will see below, the term that appears in the expressions of the cross section and the width is of the form $|d^{j}_{m,m'}(\theta)|^2$. We note that the normalization of this term gives:

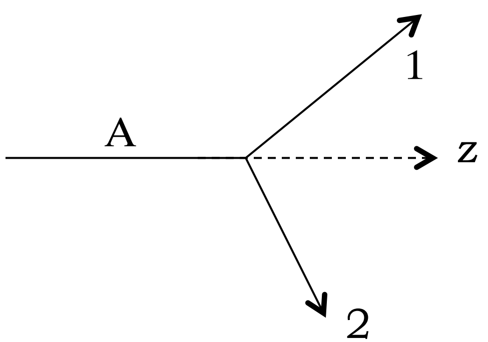
$$\frac{2j+1}{2}\int_0^{\pi} \left|d^{j}_{m,m'}(\theta)\right|^2 \sin(\theta)\, d\theta = 1.$$

Definition of axes, proceeding through the helicity formalism

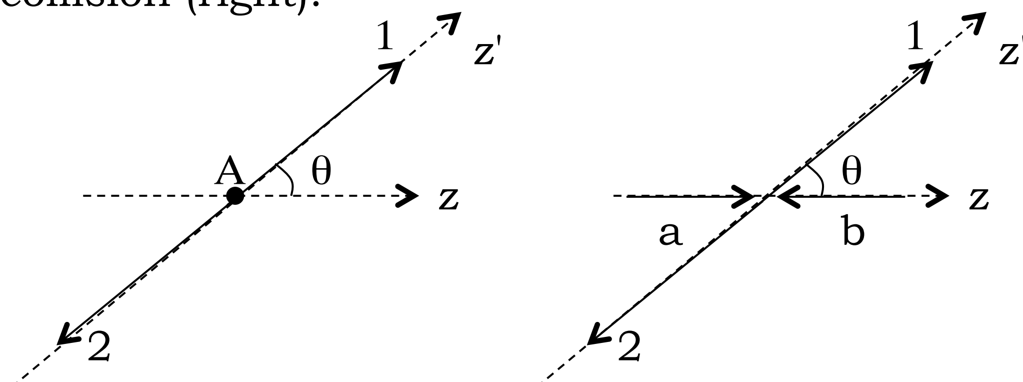
For a decay (A→1+2), the z-axis is defined as the quantification axis of the angular momentum of the particle A. Usually this axis is chosen in the direction of movement of A in the laboratory frame, and in this case the z-projection of the angular momentum simply coincides with the helicity of A. The z'–axis is defined as the axis of movement of the final state particles in the center of mass.

For a collision process a+b →1+2, the z-axis is defined as the axis of movement of the initial state particles in the center of mass. In case of a decay A →1+2 the z-axis is usually defined in a similar way, as the axis of movement of the initial-state particle in the laboratory frame. The axis z' is along the momentum of the final state particles in the center of mass, and the angle between z and z' is denoted θ.

In the laboratory frame:

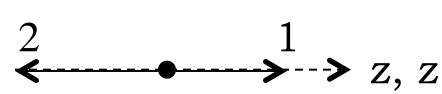


In the center of mass frame, in case of a decay (left) and collision (right):



Our aim is to determine the angular distribution (in θ) of the final-state particles. The procedure consists in first writing the state of the system for θ=0 and then rotating the obtained state by an angle θ, to finally find $\Gamma \propto \left| d^{j}_{mi\,mf}\left(\theta\right)\right|^2$

$\theta = 0$ corresponds to the configuration:

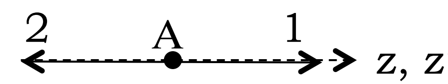


Projection of the angular momentum on an axis

What we show here, applies in particular to the case $\theta = 0$.

The conservation of the z-component of the total angular momentum J gives:

$$M_J = m_{s_1} + m_L + m_{s_2}.$$



The z-component of $\vec{L}$ is:

$$\vec{L} \cdot \hat{z} = \left(\vec{r}_{CM} \wedge \vec{p}_{CM}\right) \cdot \frac{\vec{r}_1}{r_1} \qquad (\text{for} \quad \theta = 0)$$

$$\Rightarrow \widehat{L}_z = 0 \Leftrightarrow m_L = 0.$$

Therefore: $M_J = m_{s_1} + m_{s_2}.$

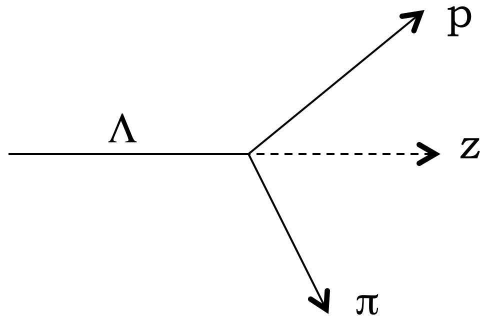
However, for $\theta = 0$ $\quad \lambda_1 = m_{s_1} \quad ; \quad \lambda_2 = -m_{s_2}$ $\quad$ following Oz

$M_J = \lambda_1 - \lambda_2$ for $\theta = 0$

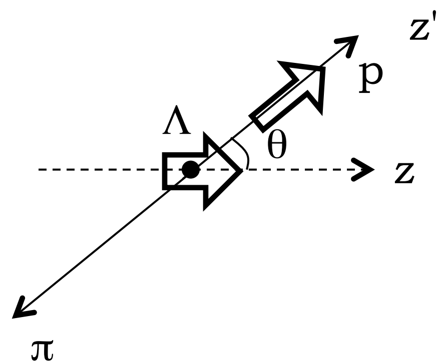
For a system of two particles, in the center of mass, the directions of the two momenta are opposite and the projection of the angular momentum on the axis defined by the momentum of one of the particles is simply the difference of the two helicities.

Illustration of the helicity formalism on the simple case: Λ→pπ

In the laboratory:



In the center of mass:



Given that the π meson is a spin-0 particle: $m_{s_\pi} = 0 \quad \Rightarrow M_J = \lambda_p = \lambda = M_\Lambda$

Furthermore: $s_\Lambda = s_P = \dfrac{1}{2}$

We now proceed to find the angular distribution (in θ) of, for instance, the final-state proton.

The probability for the proton to be scattered within the solid angle dΩ and to have a helicity λ is

$$P(\theta,\phi,\lambda) = \left|\langle\theta,\phi,\lambda|\psi_{p\pi}\rangle\right|^2 d\Omega,$$

where $|\theta,\phi,\lambda\rangle = D_z(\phi)D_y(\theta)|0,0,\lambda\rangle$

Rotation operators:

$$D_z(\phi) = e^{-i\phi/\hbar\hat{S}_z} \quad ; \quad D_y(\theta) = e^{-i\theta/\hbar\hat{S}_y}$$

(in the last expression, the helicity λ is unchanged, because $\widehat{H}$ is rotationally invariant).

Hence $P(\theta,\phi,\lambda) = \left|\langle 0,0,\lambda|\left[D_z(\phi)D_y(\theta)\right]^\dagger|\psi_{p\pi}\rangle\right|^2 d\Omega.$

To find the expression of $D_z(\phi)D_y(\theta)$, we start by:

$$D_z(\phi) = e^{-i\frac{\phi}{\hbar}\cdot S_z} = e^{-i\frac{\phi}{2}\cdot\sigma_z} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\phi}{2}\sigma_z\right)^n$$

$$\sigma_z^2 = \mathbf{1},\quad \sigma_z^{2p} = \mathbf{1},\quad \sigma_z^{2p+1} = \sigma_z$$

$$= \left[\sum_{\substack{n=0\\ n\ \text{even}}}^{\infty}\frac{1}{n!}\left(\frac{-i\phi}{2}\right)^n\right]\mathbf{1} + \left[\sum_{\substack{n=0\\ n\ \text{odd}}}^{\infty}\frac{1}{n!}\left(\frac{-i\phi}{2}\right)^n\right]\sigma_z = \cos\frac{\phi}{2}\mathbf{1} - i\sin\frac{\phi}{2}\sigma_z.$$

Therefore:

$$D_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2}\end{pmatrix}.$$

Similarly:

$$D_y(\theta) = e^{-i\theta/\hbar\sigma_y} = \cos\frac{\theta}{2}\mathbf{1} - i\sin\frac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

$$\Rightarrow D_z(\phi)D_y(\theta) = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} & e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}.$$

We next express $|\psi_{p\pi}\rangle$ in the basis $|J\,M_J\rangle$ (recall that J is the total angular momentum and M_J its projection on the quantification axis). We have $|J\,M_J\rangle = |J_\Lambda\,M_\Lambda\rangle$. Hence:

$$\frac{dP(\theta,\phi,\lambda)}{d\Omega} = \Big|\underbrace{\left\langle 0,0,\lambda\right|\left[D_z(\phi)D_y(\theta)\right]^\dagger\left|J_\Lambda M_\Lambda\right\rangle}_{A_{m\lambda}(\theta,\phi)}\Big|^2.$$

Suppose that the Λ is in the state $M_\Lambda = +1/2$, which is written as $\begin{pmatrix}1\\0\end{pmatrix}$.

We obtain:

$$D_y^\dagger(\theta)D_z^\dagger(\phi)\left|J_\Lambda,\frac{1}{2}\right\rangle = e^{i\phi/2}\cos\frac{\theta}{2}\left|J_\Lambda,+\frac{1}{2}\right\rangle - e^{i\phi/2}\sin\frac{\theta}{2}\left|J_\Lambda,-\frac{1}{2}\right\rangle$$

This allows to write:

$$A_{1/2,\lambda}(\theta,\phi) = e^{i\phi/2}\cos\frac{\theta}{2}\left\langle 0,0,\lambda \middle| \frac{1}{2},+\frac{1}{2}\right\rangle - e^{-i\phi/2}\sin\frac{\theta}{2}\left\langle 0,0,\lambda \middle| \frac{1}{2},-\frac{1}{2}\right\rangle,$$

and hence:

$$A_{1/2,\ 1/2}(\theta,\phi) = A^{+}e^{i\phi/2}\cos\frac{\theta}{2} \quad \text{with} \quad A^{+} = \left\langle 0,0,\frac{1}{2} \middle| \frac{1}{2},+\frac{1}{2}\right\rangle$$

$$A_{1/2,\ -1/2}(\theta,\phi) = -A^{-}e^{-i\phi/2}\sin\frac{\theta}{2} \quad \text{with} \quad A^{-} = \left\langle 0,0,-\frac{1}{2} \middle| \frac{1}{2},-\frac{1}{2}\right\rangle$$

The factors A^+ et A^- are *a-priori* unknown.

The probability for the proton to be scattered in the direction (θ,ϕ) is therefore:

$$\begin{cases} \lambda = +\dfrac{1}{2} & \left|A^{+}\right|^{2}\dfrac{(1+\cos\theta)}{2} \\ \lambda = -\dfrac{1}{2} & \left|A^{-}\right|^{2}\dfrac{(1-\cos\theta)}{2} \end{cases}$$

The total probability, normalized to unity is $W_{M_\Lambda=\frac{1}{2}}(\cos\theta,\phi) = \dfrac{1}{4\pi}(1+\alpha\cos\theta)$,

where $\alpha = \dfrac{\left|A^{+}\right|^{2} - \left|A^{-}\right|^{2}}{\left|A^{+}\right|^{2} + \left|A^{-}\right|^{2}}$. (The normalisation provides a constraint.)

Recapitulation – the general case

In the general case, we can write the amplitude of a decay process ($A \to 1+2$) as

$$A_H(\Omega; m, \lambda 1, \lambda 2) = \sqrt{\frac{2j+1}{4\pi}} D^j_{m,\lambda 1-\lambda 2}(\theta,\phi) A_{\lambda 1,\lambda 2}.$$

The quantity $A_{\lambda 1\, \lambda 2}$ is called the "helicity amplitude" and is characteristic of a specific helicity configuration of the final state particles. The direction of the final-state particles is given by Ω (θ, ϕ), and j is the total angular momentum of, for instance, the initial state (here, the spin of the particle A).

The corresponding width is obtained by taking the square module of the amplitude. The ϕ dependence of the rotation operators D disappear, as in the simple case above:

$$\left.\frac{d\Gamma}{d\Omega}\right|_{m,\lambda 1,\lambda 2} = \frac{2j+1}{4\pi}\left|D^j_{m,\lambda 1-\lambda 2}(\theta,\phi)\right|^2 \left|A_{\lambda 1,\lambda 2}\right|^2 = \frac{2j+1}{4\pi}\left|d^j_{m,\lambda 1-\lambda 2}(\theta)\right|^2 \left|A_{\lambda 1,\lambda 2}\right|^2.$$

Integrating this expression over the solid angle yields

$$\Gamma_{m,\lambda 1,\lambda 2} = \left|A_{\lambda 1,\lambda 2}\right|^2,$$

where we used the normalization property of the $d^j_{\lambda i,\, \lambda f}(\theta)$ functions. This expression clarifies the physical meaning of the helicity amplitude $A_{\lambda 1\, \lambda 2}$: its square module is simply the rate of decay into the helicity-specific final state, given by $\lambda_1\, \lambda_2$. The helicity amplitude describes all the physics of the problem except for the angular dependence.

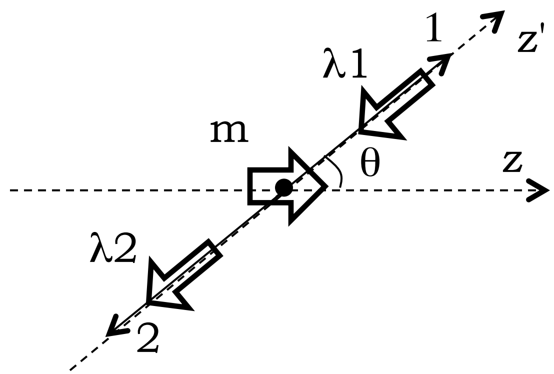
For the interaction process a+b→1+2, we can easily obtain a similar expression.

We recall that, in any case:

• *j* is the total angular momentum in the process (the spin of the decaying initial-state particle A, or of the intermediate resonance created by a+b);

• *m* is the projection of the angular momentum on the *z*-axis. It is the projection of the spin of A in the case of a decay, and λ_a-λ_b in a collision;

• λ_1-λ_2 is the projection of the final-state total angular momentum on the z'-axis.



For further reading on the topic, refer to the articles:

- CALT-68-1148 "An Experimenter's Guide to the Helicity Formalism" Jeffrey D. Richman
- "An Angular Distribution Cookbook" Rob Kutschke