From M. Schwartz :

(10.94), (10.95);

11.2, 11.2.1

12.5.2

10.4; Charge operator on Fock states

Quantization of the Dirac Field

Take Dirac's equation:

$$(i\not{\partial} - m)\psi = 0$$

We want to quantize the theory as we did for the scalar Field:

1 - Find plane-wave solutions
2 - Write the generic solution as superpositions of plane waves with arbitrary coefficients
3 - Make the coefficients into operators

Plane-wave solution of Dirac's equation

• Dirac implies Klein-Gordon:

$$(i\not{\partial} + m)(i\not{\partial} - m)\psi = 0$$

$$\parallel$$

$$0 = (i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi =$$

$$[-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu \cancel{- im\gamma^\mu\partial_\mu} \cancel{+ im\gamma^\nu\partial_\nu} - m^2]\psi =$$

$$= \left[-\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu - \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu - m^2 \right] \psi$$

$$= \Big[-\frac{1}{2} \underbrace{\{ \gamma^\mu, \gamma^\nu \}}_{2\eta^{\mu\nu}\mathbb{1}} \partial_\mu \partial_\nu - \frac{1}{2} \underbrace{[\gamma^\mu, \gamma^\nu] \partial_\mu \partial_\nu}_{= 0} - m^2 \Big] \psi$$

$$= -(\underbrace{\eta^{\mu\nu} \partial_\mu \partial_\nu}_{\Box} + m^2)\mathbb{1} \psi = 0$$

$\Rightarrow$ Dirac eq $\Rightarrow$ $(\Box + m^2)\psi_\alpha = 0 \quad \forall \alpha$

Plane wave solutions with positive frequency:

$$\psi_\alpha^{(\vec{p})}(x) = u_\alpha(\vec{p}) \underbrace{e^{-ip_0 x^0 + i\vec{p}\cdot\vec{x}}}_{e^{-ip_\mu x^\mu}} \quad \text{with} \quad p_0 = \sqrt{|\vec{p}|^2 + m^2}$$

where $u_\alpha(\vec{p})$ is a <u>constant</u> spinor.

the Dirac equation imposes a further constraint:

$$(i\cancel{\partial} - m)\psi^{(\vec{p})}(x) = (\cancel{p} - m) u(\vec{p}) e^{-ipx} = 0$$

$$\Rightarrow (\cancel{p} - m)_{\alpha\beta} u_\beta(\vec{p}) = 0$$

the general solution is :

$$\boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3}\Big(\underbrace{u(\vec{p})\, e^{-ipx}}_{\text{positive Frequency}} + \underbrace{v(\vec{p})\, e^{ipx}}_{\text{negative frequency}}\Big)}$$

where $p_0 = \sqrt{|\vec{p}|^2 + m^2}$ for both.

[For $p_0 < 0$: $\displaystyle\int_{p_0<0} \tilde{v}(\vec{p})\, e^{-ip_0x^0 + i\vec{p}\cdot\vec{x}}\, d^3p = \int \tilde{v}(\vec{p})\, e^{i|p_0|x^0 + i\vec{p}\cdot\vec{x}}\, d^3p$

$\underset{\text{change } \vec{p}\to-\vec{p}}{=} \displaystyle\int d^3p\, v(\vec{p})\, e^{ip_0x^0 - i\vec{p}\cdot\vec{x}}\, d^3p$

where $v(\vec{p}) = \tilde{v}(-\vec{p})$ $\quad p_0 > 0$ now.]

$$(\not{p} - m)\, u(\vec{p}) = 0 \qquad (\not{p} + m)\, v(\vec{p}) = 0$$

(since $\psi(x)$ is complex, $v_\alpha(\vec{p}) \neq u^*_\alpha(p)$)

So now we have to solve an equation in spinor space :

$$(\not{p} - m\mathbb{1})_{\alpha\beta}\, u_\beta(\vec{p}) = 0 \Rightarrow \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

Recall : $\sigma^\mu = (\mathbb{1}, \sigma_i)$; $\bar\sigma^\mu = (\mathbb{1}, -\sigma_i)$

. Consider first the <u>rest frame</u> : $p = (m, \vec{0})$

$$\rightarrow \begin{pmatrix} -m\mathbb{1}_2 & m\mathbb{1}_2 \\ m\mathbb{1}_2 & -m\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0 \Rightarrow \boxed{\psi = \chi}$$

$$\Rightarrow \quad \underset{\text{convenient normalization}}{u_{(m,\vec{0})} = \sqrt{m}} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

a complete set of solution is obtain with $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ there are 2 independent positive-frequency solution labeled by $s = 1, 2$

Now take $p = (E, 0, 0, p_z)$ $E = (p_z^2 + m^2)^{1/2}$

$$(\not{p} - m) = \begin{pmatrix} -m\mathbb{1} & p_0\mathbb{1} - p_z\sigma_3 \\ p_0\mathbb{1} + p_z\sigma_3 & -m\mathbb{1} \end{pmatrix} =$$

$$= \left(\begin{array}{cc|cc} -m & 0 & p_0 - p_z & 0 \\ 0 & -m & 0 & p_0 + p_z \\ \hline p_0 + p_z & 0 & -m & 0 \\ 0 & p_0 - p_z & 0 & -m \end{array}\right) \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0$$

$$\psi = \begin{pmatrix} \sqrt{p_0 - p_z} & 0 \\ 0 & \sqrt{p_0 + p_z} \end{pmatrix} \xi$$

solution:

$\xi = \xi_1, \xi_2$

$$\chi = \begin{pmatrix} \sqrt{p_0 + p_z} & 0 \\ 0 & \sqrt{p_0 - p_z} \end{pmatrix} \xi$$

Can we write this for a <u>generic</u> p_μ?

$$\sqrt{p_0 - p_z} = \sqrt{p_\mu \sigma^\mu} \quad \text{for } p_\mu = (p_0, 0, 0, p_z)$$

$$\sqrt{p_0 + p_z} = \sqrt{p_\mu \bar{\sigma}^\mu}$$

$$\Rightarrow \quad u_s = \begin{pmatrix} \sqrt{p \cdot \sigma} \; \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \; \xi_s \end{pmatrix} \qquad \xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can make this look nicer (no $\sqrt{\text{matrices}}$): notice that

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2$$

$$(p\cdot\sigma)(p\cdot\bar\sigma) = (p_\mu \sigma^\mu)(p_\nu \bar\sigma^\nu) =$$

$$= (p_0 \mathbb{1} + p_i \sigma_i)(p_0 \mathbb{1} - p_j \sigma_j) =$$

$$= p_0^2 + \cancel{p_i \sigma_i p_0} - \cancel{p_j \sigma_j p_0} - p_i p_j \sigma_i \sigma_j$$

$$= p_0^2 - \frac{1}{2} p_i p_j \underbrace{\{\sigma_i, \sigma_j\}}_{2\delta_{ij}} = p_0^2 - |\vec p|^2 = m^2$$

$$\sqrt{p\cdot\bar\sigma}\, u_s = \sqrt{p\cdot\bar\sigma} \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi_s \\ \sqrt{p\cdot\bar\sigma}\, \xi_s \end{pmatrix} = \begin{pmatrix} m\, \xi_s \\ (p\cdot\bar\sigma)\xi_s \end{pmatrix}$$

[Check Dirac:

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix} \begin{pmatrix} m\xi_s \\ (p\cdot\bar\sigma)\xi_s \end{pmatrix} = \begin{pmatrix} -m^2 \xi_s + \overbrace{(p\cdot\sigma)(p\cdot\bar\sigma)}^{m^2}\xi_s \\ m(p\cdot\bar\sigma)\xi_s - m(p\cdot\bar\sigma)\xi_s \end{pmatrix}$$

$= 0$]

Usually take the definition:

$$
\boxed{
\begin{aligned}
u_s(\vec{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\;\xi_s \end{pmatrix} \qquad s = 1,2 \\[2ex]
v_s(\vec{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta_s \end{pmatrix} \qquad s = 1,2
\end{aligned}
}
$$

$$
\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

For example for $\vec{p} = p_z$:

$$
u_1(p) = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ \sqrt{E-p_z} \\ 0 \end{pmatrix}, \quad
u_2(p) = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ \sqrt{E+p_z} \end{pmatrix}, \quad
v_1(p) = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ -\sqrt{E-p_z} \\ 0 \end{pmatrix}, \quad
v_2 = \begin{pmatrix} 0 \\ \sqrt{E+p_z} \\ 0 \\ -\sqrt{E+p_z} \end{pmatrix}
$$

Spin

take $p_z = 0$, $E = m$.

$$u_1 = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad V_1 = \sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$u_2 = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \qquad V_2 = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

Recall the spin-generators:

$$S^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

In particular:

$$S^3 = \frac{1}{2}\begin{pmatrix}1 & & & 0\\ & -1 & & \\ & & 1 & \\ 0 & & & -1\end{pmatrix}$$

$$S^3 u_1 = +\frac{1}{2}u_1 \ , \quad S^3 V_1 = +\frac{1}{2}V_1$$

$$S^3 u_2 = -\frac{1}{2}u_2 \ , \quad S^3 V_2 = -\frac{1}{2}V_2$$

$\Rightarrow$ $u_1, V_1 \Rightarrow$ spin <u>up</u> states

$u_2, V_2 \Rightarrow$ spin <u>down</u> states

Normalisation:

Consider the inner product:

$$\bar{u}_s(p)\, u_{s'}(p) = u_s^\dagger(p)\,\gamma^0 u_{s'}(p) =$$

$$= \left((\sqrt{p\cdot\sigma}\,\xi_s)^\dagger, (\sqrt{p\cdot\bar{\sigma}}\,\xi_s)^\dagger \right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{s'} \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_{s'} \end{pmatrix} =$$

$$= \left(\xi_s^\dagger, \xi_s^\dagger \right) \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & 0 \\ 0 & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix}$$

$$(p\cdot\sigma)^\dagger = (p\cdot\sigma)$$

$$= \underbrace{\xi_s^\dagger \cdot \xi_{s'}}_{\delta_{ss'}}\, m + \underbrace{\xi_s^\dagger \xi_{s'}}_{\delta_{ss'}}\, m = 2m\,\delta_{ss'}$$

Because we chose a orthonormal basis for ξ_s

$$\Rightarrow \boxed{\begin{cases} \bar{u}_s(p)\, u_{s'}(p) = 2m\,\delta_{ss'} \\ \bar{v}_s(p)\, v_{s'}(p) = -2m\,\delta_{ss'} \\ \bar{u}_s(p)\, v_{s'}(p) = \bar{v}_s(p)\, u_{s'}(p) = 0 \end{cases}}$$

Other useful relations: (exercise)

$$u_s^\dagger(\vec{p})\, u_{s'}(\vec{p}) = 2E\,\delta_{ss'} \qquad v_s^\dagger(\vec{p})\, u_{s'}(-\vec{p}) = 0$$

$$v_s^\dagger(\vec{p})\, v_{s'}(\vec{p}) = 2E\,\delta_{ss'} \qquad u_s^\dagger(\vec{p})\, v_{s'}(-\vec{p}) = 0$$

Spin sums

take the expression $u_s(p)\bar{u}_s(p)$

this is a 4×4 matrix (no contraction):

$$\left(u_s(p)\right)_\alpha \left(\bar{u}_s(p)\right)_\beta \equiv \mathcal{M}^{(s)}_{\alpha\beta} \qquad \alpha,\beta = 1,2,3,4$$

now sum over $s = 1,2$

$$\sum_{s=1}^{2} u_s(p)_\alpha \bar{u}_s(p)_\beta = p_\mu \gamma^\mu_{\alpha\beta} + m\mathbb{1}_{\alpha\beta}$$

(exercise)

[This comes from the completeness relation of the ξ_s:

$$\sum_{s=1}^{2} \xi_{s\,i}\, \xi_{s\,j} = \delta_{ij} \ ]$$

So:

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m$$

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

(4×4 matrix identities)

Quantum Dirac Field

the general solution of Dirac's eq is:

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[c_{s,\vec{p}}\, u_s(p) e^{-ipx} + d^*_{s,\vec{p}}\, v_s(p) e^{ipx} \right]$$

convention

where now $u_s(p)$ and $v_s(p)$ have fixed normalization and we have included arbitrary coefficients c, d^*.
(if the field were real, $d_{p,s} = c_{p,s}$)
(these coefficients are numbers as the spinorial index is carried by u and v)

__Quantization__ : $c_{s,\vec{p}} \longrightarrow \hat{c}_{s,\vec{p}}$

$d^*_{s,\vec{p}} \longrightarrow \hat{d}^{\dagger}_{s,\vec{p}}$

$\hat{c}_{s,\vec{p}}$ is an __annihilation__ operator
(it multiplies positive frequency modes)

$\hat{d}^{\dagger}_{s,\vec{p}}$ is a __creation__ operator
(it multiplies negative frequency modes)

(think a_p and $\hat{a}^{\dagger}_p$ for the real scalar).

Fock space

Vacuum state : $|0\rangle$ $\qquad \hat{c}_{s,\vec{p}}|0\rangle = 0$

$\hat{d}_{s,\vec{p}}|0\rangle = 0$

1-particle states :

$$|\vec{p}, s\rangle = \begin{cases} \hat{c}^{+}_{s,\vec{p}}|0\rangle \\ d^{+}_{s,\vec{p}}|0\rangle \end{cases}$$

$s=1$: spin $1/2$; $s=2$: spin $-1/2$

- there are 2 kinds of particles for each spin, those created by the $\hat{c}^{+}$, and those created by the $\hat{d}^{+}$;
- they have the same <u>mass</u> since for both $p^2 = m^2$.
- As we will see, they have opposite <u>charge</u> with respect to a global $U(1)$ (which eventually can be coupled to E.M.)

- this is the same as for the complex scalar (there was also 2 kinds of oscillators, and a $U(1)$ symmetry).

- Introduce a label q to tell $c^\dagger$ from $d^\dagger$: $q = +$ or $-$

$$|\vec{p}, s, q\rangle : \quad |\vec{p}, s, +\rangle = c^\dagger_{\vec{p},s}|0\rangle$$
$$|\vec{p}, s, -\rangle = d^\dagger_{\vec{p},s}|0\rangle$$

- Multi-particle states :

$$|\vec{p}_1, s_1, +; \dots \vec{p}_n, s_n, +; \vec{k}_1, s_1, -, \dots \vec{k}_m, s_m, -\rangle =$$

$$\mathcal{N} \underbrace{c^\dagger_{\vec{p}_1, s_1} \dots c^\dagger_{\vec{p}_n, s_n}}_{\substack{n \text{ particle with momenta} \\ \text{and spin } \vec{p}_1, s_1 \dots \vec{p}_n, s_n}} \underbrace{d^\dagger_{\vec{k}_1, r_1} \dots d^\dagger_{k_n, r_n}}_{\substack{m \text{ anti-particles with} \\ \text{momenta } \vec{k}_1, r_1 \dots \vec{k}_m, r_m}} |0\rangle$$

Commutation (?) rules

Let us construct the Hamiltonian (the energy operator) on the Fock space.

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

Energy-momentum tensor:

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^\mu \psi} \partial_\nu \psi - \eta_{\mu\nu} \mathcal{L}$$

$$\frac{\partial}{\partial \partial^\mu \psi} \bar{\psi}(i\gamma_\rho \partial^\rho \psi) = i\bar{\psi}\gamma_\rho \delta^\rho_\mu = i\bar{\psi}\gamma_\mu$$

$$\Rightarrow T_{\mu\nu} = \bar{\psi} i \gamma_\mu \partial_\nu \psi - \eta_{\mu\nu}\bar{\psi}(i\gamma^\rho\partial_\rho - m)\psi$$

The energy density is:

$$T_{00} = i\bar{\psi}\gamma_0\partial_0\psi - \bar{\psi}(i\not{\partial} - m)\psi$$

$$= \cancel{i\bar{\psi}\gamma_0\partial_0\psi} - \bar{\psi}(\cancel{i\gamma^0\partial_0} - i\gamma^i\partial_i - m)\psi$$

$$= \bar{\psi}(i\gamma^i\partial_i + m)\psi$$

Since we are going to expand this on solutions of the Dirac equation:

$$(i\gamma^i \partial_i + m)\psi = (i\gamma^i\partial_i - i\gamma^0\partial_0 + i\gamma^0\partial_0 + m)\psi$$

$$= [i\gamma^0\partial_0 - \cancel{(i\not{\partial} - m)}]\psi$$

$$\Rightarrow \qquad \underline{T_{00} = i\bar{\psi}\gamma^0\partial_0\psi}$$

Let us write the Total energy operator:

$$\hat{E} = \int d^3x\, T_{00} = i\int d^3x\, \bar{\psi}(x)\gamma^0\partial_0\psi(x)$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_s \left(e^{-ipx}u_s(\vec{p})\hat{c}_{s,\vec{p}} + e^{+ipx}v_s(\vec{p})d^\dagger_{s\vec{p}}\right)$$

$$\bar{\psi}(x) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\sum_{s'}\left(e^{iqx}\bar{u}_{s'}\hat{c}^\dagger_{s',\vec{q}} + e^{-iqx}\bar{v}_{s'}(\vec{q})d_{s',\vec{q}}\right)$$

Plug in:

$$E = i\int d^3x \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}} \times \sum_{s,s'}\left(e^{iqx}\bar{u}_{s'}(\vec{q})c^\dagger_{s'\vec{q}} + e^{-iqx}\bar{v}_{s'}(\vec{q})d_{s'\vec{q}}\right)\gamma^0\partial_t\left(e^{-ipx}u_s(\vec{p})c_{s\vec{p}} + e^{ipx}v_s(\vec{p})d^\dagger_{s\vec{p}}\right)$$

$$= i\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{ss'}\int d^3x\Big\{e^{i(q-p)x}(-ip_0)\bar{u}_{s'}(\vec{q})\gamma_0 u_s(\vec{p})c^\dagger_{s'\vec{q}}c_{sp} + e^{-i(q-p)x}\,ip_0\,\bar{v}_{s'}(\vec{q})\gamma_0 v_s(p)d_{s'\vec{q}}d^\dagger_{s\vec{p}} +$$

$$+ e^{i(q+p)x}(ip_0)\bar{u}_{s'}(\vec{q})\gamma_0 v_s(\vec{p})c^\dagger_{s'\vec{q}}d^\dagger_{s\vec{p}} + e^{-i(q+p)x}(-ip_0)\bar{v}_{s'}(\vec{q})\gamma_0 u_s(p)d_{s'\vec{q}}c_{sp}\Big\}$$

$$\int d^3x\, e^{i(q-p)x} = (2\pi)^3\delta^3(\vec{p}-\vec{q})\, \underbrace{\cancel{e^{i(p_0-q_0)}}}_{=1}$$

$$\int d^3x\, e^{i(p+q)x} = (2\pi)^3\delta^3(\vec{p}+\vec{q})\, e^{i(p_0+q_0)}$$

- the $\bar{u}v$ and $\bar{v}u$ terms contain:

$$\bar{u}_{s'}(\vec{q})\gamma_0 v_s(-\vec{q}) = u^\dagger_{s'}(\vec{q})\, v_s(-\vec{q}) = 0$$

$$\bar{v}_{s'}(\vec{q})\gamma_0 u_s(-\vec{q}) = v^\dagger_{s'}(\vec{q})\, u_s(-\vec{q}) = 0$$

$\Rightarrow$ they vanish

- the $\bar{u}u$ and $\bar{v}v$ terms are:

$$\cancel{i}\int \frac{d^3p\, \cancel{(2\pi)^3}}{(2\pi)^3\cancel{(2\pi)^3}\cancel{\sqrt{2\omega_p 2\omega_p}}} \sum_{s,s'}\Big[(-\cancel{i}p_0)\, \underbrace{u^\dagger_{s'}(\vec{p})u_s(\vec{p})}_{\delta_{s's}2\omega_p} c^\dagger_{s'p}c_{sp} + (+\cancel{i}p_0)\, \underbrace{v^\dagger_{s'}(\vec{p})v_s(\vec{p})}_{\delta_{s's}2\omega_p}\, d_{s'\vec{p}}\, d^\dagger_{s\vec{p}}\Big]$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \cancel{\frac{1}{2}}\, 2\omega_{\vec{p}} \sum_s \left(c^\dagger_{s\vec{p}}c_{sp} - d_{s\vec{p}}\, d^\dagger_{s\vec{p}}\right)$$

!!

Now we want to write it in terms of the number operators

$$\begin{cases} N_{\vec{p},s+} = c^\dagger_{\vec{p},s}c_{\vec{p}s} \\ N_{\vec{p},s-} = d^\dagger_{\vec{p}s}d_{\vec{p}s} \end{cases}$$

$N_{\vec{p},s+}$ = # of particles (created by $c^\dagger$)

$N_{\vec{p},s-}$ = # of "anti" particles (created by $d^\dagger$)

$$E = \int \frac{d^3p}{(2\pi)^3} \sum_s \omega_{\vec{p}} \left(c^\dagger_{\vec{p}s} c_{\vec{p}s} - d_{ps} d^\dagger_{ps} \right)$$

Now suppose we apply commutation rules:

$$[d_{p,s}\, d^\dagger_{q,s'}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\, \delta_{ss'}$$

$$\Rightarrow d_{ps} d^\dagger_{ps} = d^\dagger_{ps} d_{ps} - \underbrace{(2\pi)^3 \delta(\vec{0})}_{\text{Volume of space}}$$

$$\Rightarrow E = \int \frac{d^3p}{(2\pi)^3} \sum_s \omega_p \left(\hat{N}^{(+)}_{\vec{p},s} - \hat{N}^{(-)}_{\vec{p}s} \right) - \underbrace{\mathcal{E}_0 V}_{\text{infinite zero-point energy}}$$

$\Rightarrow$ (−) particles contribute a negative sign to the energy

$\Rightarrow$ the Hamiltonian is unbounded below

$\Rightarrow$ the engy can be lowered arbitrarily by producing (−)-type particles

$\Rightarrow$.) $|0\rangle$ is not the ground state,

..) There is no ground state
(states with more and more antiparticles have lower and lower engy).

$\Rightarrow$ We cannot impose CCR.

Solution: we must impose

$$d_{sp} d^{+}_{s'q} \; (+) \; d^{+}_{s'\bar{q}} d_{sp} = (2\pi)^3 \delta^{(3)}(p-\vec{q}) \delta_{ss'}$$

or

$$\left\{ d_{s\vec{p}}, d^{+}_{s'\vec{q}} \right\} = (2\pi)^3 \delta^{(3)}(p-\vec{q}) \delta_{ss'}$$

$$\left\{ c_{s\vec{p}}, c^{+}_{s'\vec{q}} \right\} = (2\pi)^3 \delta^{(3)}(p-\vec{q}) \delta_{ss'}$$

Anti-commutation relation.

Fermi Statistics

the anti-commutator has deep consequences.

take a one-particle state:

$$c^{+}_{p,s} |0\rangle = |\vec{p}, s\rangle = |1_{\vec{p},s}\rangle$$

Now try to add another particle with the same momentum and spin:

$$|2_{\vec{p},s}\rangle = c^{+}_{\vec{p},s} |1_{\vec{p},s}\rangle = c^{+}_{\vec{p},s}\, c^{+}_{\vec{p},s} |0\rangle$$

But the c's anti-commute

$$= - c^{+}_{\vec{p}s}\, c^{+}_{ps} |0\rangle$$

$\Rightarrow$

In other words the c^{+} satisfy:

$$\left(c^{+}_{\vec{p}s}\right)^2 = 0$$

$\Rightarrow$ this state is zero (not $|0\rangle$, 0!)

$\Rightarrow$ there can be no two particles with the same quantum numbers (Pauli EP)

U(1) - Invariance

$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under the internal U(1):

$$\psi \longrightarrow e^{-i\alpha}\psi \qquad \alpha \in \mathbb{R} \text{ constant}$$

indeed $\bar{\psi} = \psi^\dagger \gamma^0 \longrightarrow (e^{-i\alpha}\psi)^\dagger \gamma^0 = e^{i\alpha}\bar{\psi}$

and $\bar{\psi}(i\partial\!\!\!/ - m)\psi \to \cancel{e^{i\alpha}}\,\cancel{e^{-i\alpha}}\,\bar{\psi}(i\partial\!\!\!/ - m)\psi$

<u>Nöther current</u>: take infinitesimal transformation:

$$\psi \to e^{-i\delta\alpha}\psi \simeq \psi - i\delta\alpha\,\psi = \psi + \delta\psi \Rightarrow \underline{\delta\psi = -i\delta\alpha\psi}$$

$$J_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \psi}\frac{\delta\psi}{\delta\alpha} = i\bar{\psi}\gamma_\mu(-i\delta\psi) = \bar{\psi}\gamma_\mu\psi$$

the associated charge is:

$$Q = \int d^3x\, J_0 = \int d^3x\, \bar{\psi}\gamma_0\psi$$

We can write this in terms of the creation/annihilation operators:

$\Rightarrow$ same calculation as before, except there is no ∂_t

$$\Rightarrow Q = \int \frac{d^3p \quad (2\pi)^3}{(2\pi)^3 (2\pi)^3 \sqrt{2\omega_p 2\omega_p}} \sum_{s,s'} \Big[\cancel{(-iP_0)} \underbrace{u^\dagger_{s'}(\vec{p}) u_s(\vec{p})}^{\delta_{s's} 2\omega_p} C^\dagger_{s'p} C_{sp} + \cancel{(+iP_0)} \underbrace{v^\dagger_{s'}(\vec{p}) v_s(p)}_{\delta_{s's} 2\omega_p} d_{s'\vec{p}} d^\dagger_{s\vec{p}} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(c^\dagger_{s\vec{p}} c_{s\vec{p}} + d_{s\vec{p}} d^\dagger_{s\vec{p}} \right)$$

anticommute and discard ∞ contribution

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(c^\dagger_{s\vec{p}} c_{s\vec{p}} - d^\dagger_{s\vec{p}} d_{s\vec{p}} \right)$$

$$\left(N^{(+)}_{s\vec{p}} - N^{(-)}_{s\vec{p}} \right)$$

Q counts (# particles − # antiparticles)!

$\Rightarrow$ If particles have charge $-e$ (electrons) antiparticles have charge $+e$ (positrons)

$J_\mu = \bar{\psi} \gamma_\mu \psi$ ≡ electron number current

$J^{(e)}_\mu = e \bar{\psi} \gamma_\mu \psi$ ≡ electric current e.g. take a state with n particles and m anti-particles:

$$|\psi\rangle = |\vec{p}_1, s_1, + ; \ldots \vec{p}_n, s_n + ; \vec{q}_1, s_1, - ; \ldots ; \vec{q}_m, s_m - \rangle$$

$$= c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger |0\rangle$$

$$Q|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \left(c_{\vec{k}}^\dagger c_{\vec{k}} - d_{\vec{k}}^\dagger d_{\vec{k}} \right) |\psi\rangle =$$

$$= \int \frac{d^3k}{(2\pi)^2} c_{\vec{k}}^\dagger c_{\vec{k}} c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger |0\rangle$$

$$- \int \frac{d^3k}{(2\pi)^3} d_{\vec{k}}^\dagger d_{\vec{k}} c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger |0\rangle =$$

$$= \int \frac{d^3k}{(2\pi)^3} \Big\{ \left[c_k^\dagger c_k, c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger \right] |0\rangle + c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger \cancel{c_k^\dagger c_k |0\rangle} \Big\}$$

$$- \int \frac{d^3k}{(2\pi)^3} \Big\{ \left[d_k^\dagger d_k, c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger \right] |0\rangle + c_1^\dagger \ldots c_n^\dagger d_1^\dagger \ldots d_m^\dagger \cancel{d_k^\dagger d_k |0\rangle}$$

So we need the commutator of $N_k^+ = c_k^\dagger c_k$

with $c_1^\dagger \cdots c_n^\dagger d_1^\dagger \cdots d_m^\dagger$

$$[c_k^\dagger c_k, c_p^\dagger] = c_k^\dagger c_k c_p^\dagger - c_p^\dagger c_k^\dagger c_k = c_k^\dagger c_k c_p^\dagger + c_k^\dagger c_p^\dagger c_k$$

$$= c_k^\dagger \cancel{c_k} c_p - c_k^\dagger \cancel{c_k} c_p + (2\pi)^3 \delta^2(p-k) c_p^\dagger$$

$$[c_k^\dagger c_k, d_p^\dagger] = c_k^\dagger c_k d_p^\dagger - d_p^\dagger c_k^\dagger c_k = c_k^\dagger c_k d_p^\dagger + c_k^\dagger d_p^\dagger c_k = c_k^\dagger c_k d_p^\dagger - c_k^\dagger c_k d_p^\dagger = 0$$

$$[N_{\vec{k}}^{(+)}, c_{\vec{p}}^\dagger] = (2\pi)^3 \delta^3(\vec{p}-\vec{k}) c_p^\dagger$$

$$[N_{\vec{k}}^{(-)}, d_{\vec{q}}^\dagger] - (2\pi)^3 \delta^{(3)}(p-\vec{q}) d_q^\dagger$$

$$[N_k^{(+)}, d_q^\dagger] = [N_k^{(-)}, c_p^\dagger] = 0$$

So if one has 2:

$$[N_k^{(+)}, c_{p_1}^\dagger c_{p_2}^\dagger] = c_{p_1}^\dagger [N_k^+, c_{p_2}^+] + [N_k^+, c_{p_1}^\dagger] c_{p_2}^\dagger$$

$$= c_{p_1}^\dagger c_{p_2}^\dagger (2\pi)^3 \delta^3(k-p_2) + (2\pi)^3 \delta^{(3)}(k-p_1) c_{p_1}^\dagger c_{p_2}^\dagger$$

$$= \{(2\pi)^3 \delta^{(3)}(k-p_2) + (2\pi)^3 \delta^3(k-p_1)\} c_{p_1}^\dagger c_{p_2}^\dagger$$

$$Q|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{i=1}^{n} (2\pi)^3 \delta^3(\vec{k}-\vec{p}_i) c_1^\dagger \cdots c_n^\dagger d_1^\dagger \cdots d_n^\dagger |0\rangle$$

$$- \int \frac{d^3k}{(2\pi)^3} \sum_{i=1}^{m} (2\pi) \delta^3(\vec{k}-\vec{q}_i) c_1^\dagger \cdots c_n^\dagger d_1^\dagger \cdots d_n^\dagger |0\rangle$$

each δ-function gives one and kills the $\int d^3k/(2\pi)^3$

$$= n\, c_1^\dagger \cdots c_n^\dagger d_1^\dagger \cdots d_n^\dagger |0\rangle - m\, c_1^\dagger \cdots c_n^\dagger d_1^\dagger \cdots d_n^\dagger |0\rangle$$

$$\Rightarrow \quad \boxed{Q|\psi\rangle = (\underbrace{n}_{\#\text{particles}} - \underbrace{m}_{\#\text{antiparticles}})|\psi\rangle}$$

the <u>electric charge</u> is

$$Q_e = -eQ = -e n_e + e n_{\text{positrons}}$$