Q.E.D. (Quantum ElectroDynamics)

Building Blocks:

- A spin 1/2 Dirac fermion ψ, (electron) massive, charged
- A spin 1 field, massless, A_μ (photon), neutral
- An interaction which conserves electric charge, and comes from gauge-invariance under:

$$\begin{cases} \psi \rightarrow e^{i\alpha(x)q}\,\psi \\ A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x) \end{cases}$$

$\alpha(x)$ Real function

q: gauge coupling

These requirements (plus the absence of coupling whose mass-dimension is negative) fix uniquely the Lagrangian to be:

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{D} - m)\,\psi$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad D_\mu \psi \equiv \partial_\mu \psi - iqA_\mu \psi$$

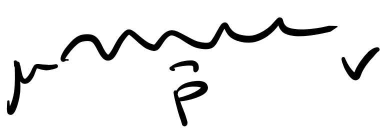
We can identify a <u>quadratic</u> part describing free spin-1/2 and spin-1 particles, and an interaction term:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \longrightarrow \text{Free photons}$$

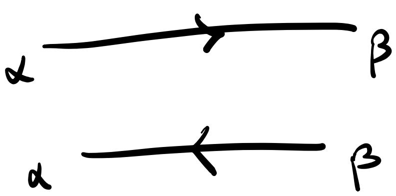
$$+ \bar{\psi}(i\not{\partial} - m)\psi \longrightarrow \text{Free electrons}$$

$$+ q A_\mu \bar{\psi} \gamma^\mu \psi \longrightarrow \text{Photon - electron interaction}$$

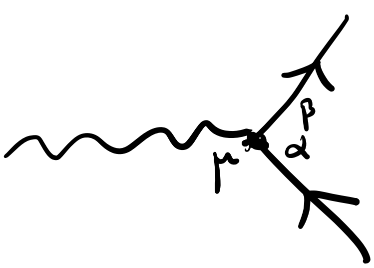
<u>Feynman Rules</u>:



$$= \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$$ Photon propagator in Feynman gauge



Dirac Field propagator (coming soon)



$$iq\gamma^\mu_{\alpha\beta}$$ interaction vertex

Dirac spinor propagator

$\alpha, \beta = 1,2,3,4$

$$\mathcal{L}_D^{free} = \bar{\Psi}_\alpha \underbrace{\left(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\mathbb{1}_{\alpha\beta} \right)}_{O_{\alpha\beta}} \Psi_\beta$$

Propagator: $S_{\alpha\beta} = O^{-1}_{\alpha\beta} = \left(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\mathbb{1}_{\alpha\beta} \right)^{-1}$

momentum space $i\partial_\mu \to p_\mu$

$$S_{\alpha\beta}(p) = \left(\gamma^\mu_{\alpha\beta} p_\mu - m\mathbb{1}_{\alpha\beta} \right)^{-1}$$

$\hookrightarrow$ 4×4 matrix in spinor space

Claim:

$$\boxed{\left(\gamma^\mu p_\mu - m\mathbb{1} \right)^{-1}_{\alpha\beta} = \frac{\gamma^\mu_{\alpha\beta} p_\mu + m\mathbb{1}_{\alpha\beta}}{p^2 - m^2}}$$

Proof = exercise. (use $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$)

$\gamma^\mu p_\mu \equiv \not{p}$

$$\boxed{S_F(p) = \frac{\not{p} + m\mathbb{1}}{p^2 - m^2 + i\epsilon}}$$

4

In position space:

$$i S_F(x-y)_{\alpha\beta} = \langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(y)) | 0 \rangle$$

$$T(\psi(x)\psi(y)) = \begin{cases} \psi(x)\psi(y) & \text{if } x^0 > y^0 \\ -\psi(y)\psi(x) & \text{if } y^0 > x^0 \end{cases}$$

particle going from y to x

= antiparticle going from x to y

$$\langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(y)) | 0 \rangle$$

creates a particle
destroys an anti-particle

particle goes from x to y

$$\langle 0 | T(\bar{\psi}_\alpha(x) \psi_\beta(y)) | 0 \rangle$$

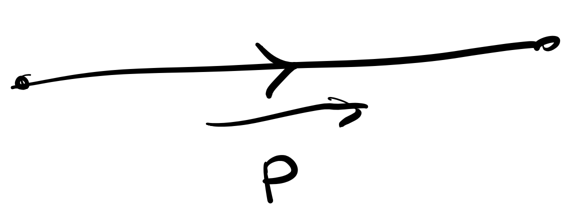
creates an antiparticle at y and destroys a particle

=

anti-particle going from y to x

=

$$-\langle 0 | T(\psi(y) \bar{\psi}(x)) | 0 \rangle$$



$$= i \frac{\cancel{p} + m}{p^2 - m^2 + i\epsilon}$$

LSZ for spinors

Recall: $(i\not{\partial} - m)\psi = 0$ Free Dirac equation

has 4 independent solutions: (in Fourier space)

- Positive frequency:

$$u^s(\vec{p})\, e^{-ip\cdot x} \qquad u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$

where $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad s = 1,2$

- Negative frequency:

$$v^s(\vec{p})\, e^{ip\cdot x} \qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$$

$\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad s = 1,2$

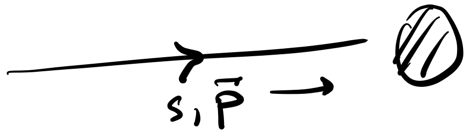
$$u^{\dagger r}(\vec{p})\, u^s(\vec{p}) = 2p_0\,\delta^{rs}$$

$$\sum_{s=1}^{2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \not{p} + m\,, \quad \sum_{s=1}^{2} v^s(\vec{p})\,\bar{v}^s(\vec{p}) = \not{p} - m$$

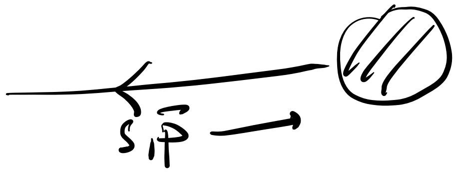
$$\langle f | S | i \rangle$$

$|f\rangle = |\, k^+$ outgoing particles and k^- outgoing antiparticles $\rangle$

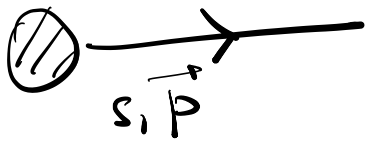
$|i\rangle = |\, e^+$ incoming particles and e^- incoming antiparticles $\rangle$

ingoing particle: 

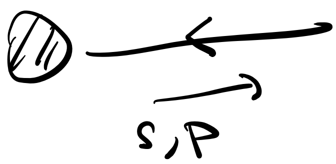
$\leftrightarrow u^s(\vec p)$

ingoing antiparticle: 

$\leftrightarrow \bar v^s(\vec p)$

outgoing particle: 

$\leftrightarrow \bar u^s(\vec p)$

outgoing antiparticle: 

$\rightarrow v^s(\vec p)$

$$\langle f|S|i\rangle = u^{s_1}_{\alpha_1}(p^i_{1^+}) \cdots u^{s_{e^+}}_{\alpha_{e^+}}(p^i_{e^+})\, \bar v^{s_1}_{\alpha_1}(p^i_{1^-}) \cdots \bar v^{s_{e^-}}_{\alpha_{e^-}}(p^i_{e^-}) \times$$

$$\times\, \bar u^{s_1}_{\alpha_1}(p^f_{1^+}) \cdots \bar u^{s_{k^+}}_{\alpha_{k^+}}(p^f_{k^+})\, v^{s_1}_{\alpha_1}(p^f_{1^-}) \cdots v^{s_{k^-}}_{\alpha_{k^-}}(p^f_{k^-}) \times$$

$$\times \left\langle \psi_{\alpha_1}(p^i_1) \cdots \psi_{\alpha_{e^-}}(p^i_{e^-})\, \psi_{\alpha_1}(p^f_{1^+}) \cdots \psi_{\alpha_{k^+}}(p^f_{k^+})\, \bar\psi_{\alpha_1}(p^i_{1^+}) \cdots \bar\psi_{\alpha_{e^+}}(p^i_{e^+})\, \bar\psi_{\alpha_1}(p^f_{1^-}) \cdots \bar\psi_{\alpha_{k^-}}(p^f_{k^-}) \right\rangle$$

$$\Big[\underbrace{\prod(\not p - m)}_{\text{spinor}}\ \underbrace{\prod(\not p + m)}_{\text{conjugate spinors}}\Big] \longrightarrow \text{cancel external propagators}$$

Feynman Rules for QED S-Matrix elements

$\longrightarrow \quad \dfrac{i(\not p + m)}{p^2 - m^2 + i\epsilon}$

$\sim\sim\sim \quad -\dfrac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$

$(-ie) A_\mu \bar\psi \gamma^\mu \psi$

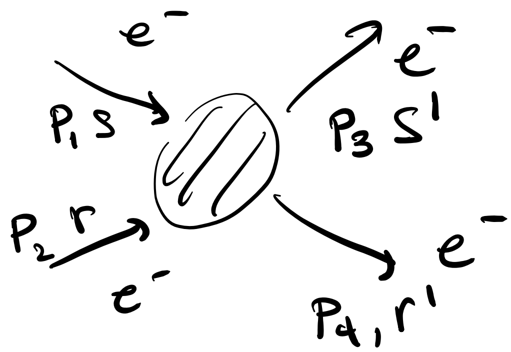
ingoing particle : $s, \vec p \to$ $\leftarrow u^s(\vec p)$

ingoing antiparticle $s, \vec p \to$ $\leftarrow \bar v^s(\vec p)$

outgoing particle $s, \vec p$ $\leftarrow \bar u^s(\vec p)$

outgoing antiparticle $s, \vec p$ $\to v^s(\vec p)$

$$e^- e^- \longrightarrow e^- e^-$$



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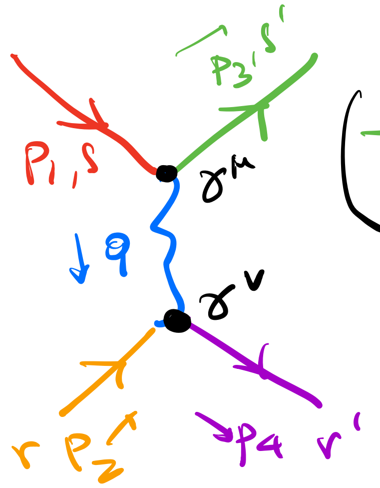


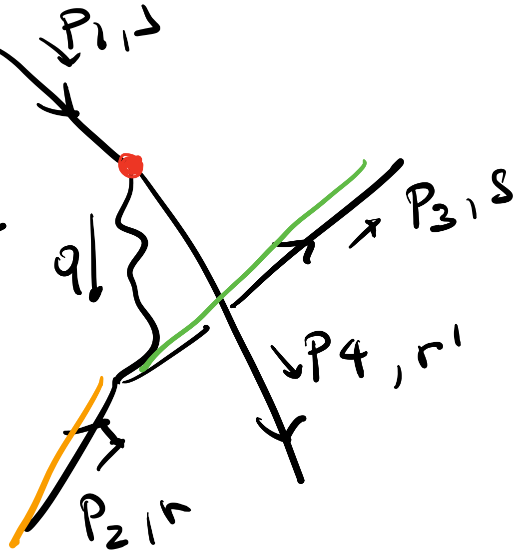
$$\left(\bar v^{s'}_\alpha(p_3)\, ie\gamma^\mu_{\alpha\beta}\, u^s_\beta(p_1)\right)\frac{\eta_{\mu\nu}}{q^2 + i\epsilon}\left(\bar u^{r'}_{\alpha'}(p_4)\,\gamma^\nu_{\alpha'\beta'}\, ie\, u^r_{\beta'}(p_2)\right)$$

$$J^\mu \qquad \frac{\eta_{\mu\nu}}{q^2} \qquad K^\nu$$

$q = p_1 - p_3$

$$\left(\Rightarrow \text{scalar}(p_1\cdot p_2,\, p_3\cdot p_4,\, s, r, s', r') \right)$$

$+$



$$(-ie)^2\, \bar{u}^{r'}_{\alpha}(p_4)\,\gamma^{\mu}_{\alpha\beta}\, u^{s}_{\beta}(p_1)\,\frac{\eta_{\mu\nu}}{q^2+i\epsilon}\,\bar{u}^{s'}_{\alpha'}(p_3)\,\gamma^{\nu}_{\alpha'\beta'}\, u^{r}_{\beta'}(p_2)$$

$$q = p_1 - p_4$$

$$e^+ e^- \longrightarrow \mu^+ \mu^-$$

- QED : $A_\mu , \psi_e , \psi_\mu$

$$\mathcal{L} = \underbrace{\bar{\psi}_e (i \overset{A_\mu}{\not{D}} - m_e) \psi_e}_{e^+, e^-} + \underbrace{\bar{\psi}_\mu (i \overset{A_\mu \;\; \mu^+, \mu^-}{\not{D}} - m_\mu) \psi_\mu}_{} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{photons}}$$

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu + i e A_\mu)$$

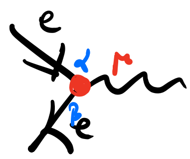
$$(q = -e)$$

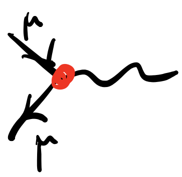
m_e = mass of electron

m_μ = mass of muon

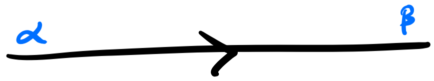
e = |charge of electron/muon|

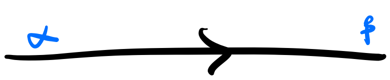
Interaction:

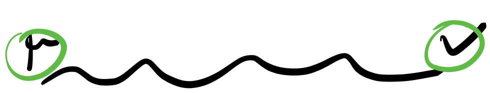
$e \bar{\psi}_e \gamma^\mu \psi A_\mu$:  $-ie\gamma^\mu_{\alpha\beta}$

$e \bar{\psi}_\mu \gamma^\mu \psi A_\mu$:  $-ie\gamma^\mu$

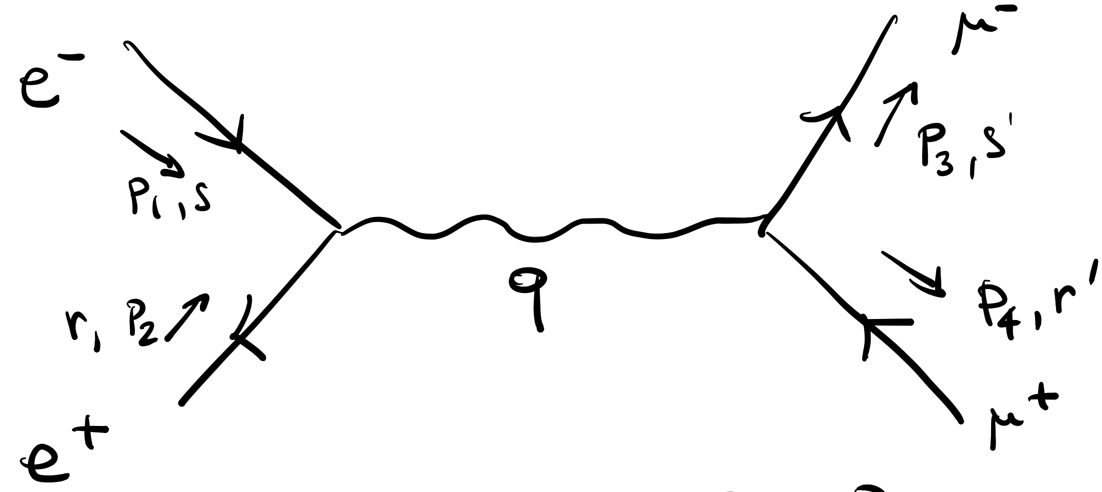
Propagators

e  $i \dfrac{(\not{p} + m_e)_{\alpha\beta}}{p^2 - m_e^2 + i\epsilon}$

μ  $i \dfrac{(\not{p} + m_\mu)_{\alpha\beta}}{p^2 - m_\mu^2 + i\epsilon}$

A_μ  $-i \dfrac{\eta^{\mu\nu}}{p^2}$

$\langle 0 | A_\mu A_\nu | 0 \rangle$



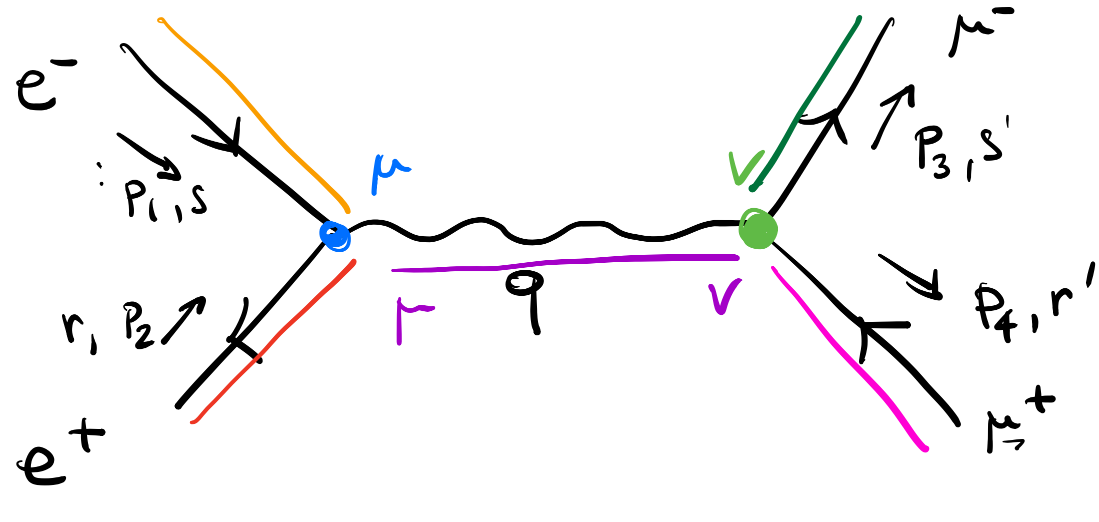
$$q = p_1 + p_2 = p_3 + p_4$$

(momentum conservation at each vertex)

$\Rightarrow$ overall momentum conservation

$(p_1 + p_2 - p_3 - p_4 = 0)$

s-channel



- ingoing particle : $s, \vec{p}$ — $u^s(p)$
- ingoing antiparticle $s, \vec{p}$ — $\bar{v}^s(\vec{p})$
- outgoing particle $s, \vec{p}$ — $\bar{u}^s(\vec{p})$
- outgoing antiparticle $s, \vec{p}$ → $v^s(\vec{p})$

$$A(p_1, p_2, p_3, p_4; s, s', r, r') =$$

$$\bar{v}^r_e(p_2)(-ie\gamma^\mu)u^s_e(p_1)\frac{(-i\eta_{\mu\nu})}{q^2+i\epsilon}\bar{u}^{s'}_\mu(p_3)(-ie\gamma^\nu)v^{r'}_\mu(p_4)$$

$|A|^2$ enters the cross section

$$= A^\dagger A \qquad (\bar{\psi}\gamma^\mu\varphi)^\dagger = \bar{\varphi}\gamma^\mu\psi$$

$$A^\dagger = i\frac{e^2}{q^2}\left[\bar{v}^{r'}(p_4)\gamma_\mu u^{s'}(p_3)\right]\left[\bar{u}^s(p_1)\gamma^\mu v^r(p_2)\right]$$

$$A = -i\frac{e^2}{q^2}\left[\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right]\left[\bar{v}^r(p_2)\gamma^\nu u^s(p_1)\right]$$

$$|A|^2(p's, \text{spins}) = A^\dagger A = \frac{e^4}{(s)^2} \times$$

$$\left[\bar{v}^{r'}(p_4)\gamma_\mu u^{s'}(p_3)\right]\left[\bar{u}^s(p_1)\gamma^\mu v^r(p_2)\right]\left[\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right]\left[\bar{v}^r(p_2)\gamma^\nu u^s(p_1)\right]$$

$(p_1+p_2)^2 = s$ Mandelstam variable s

= center of mass energy

- Suppose you cannot measure the spin of the *final* states

$$\Rightarrow \text{ Total } |A|^2 = |A|^2_{\substack{s'=\uparrow \\ r'=\uparrow}} + |A|^2_{\substack{s'=\uparrow \\ r'=\downarrow}} + |A|^2_{\substack{s'=\downarrow \\ r'=\downarrow}} + |A|^2_{\substack{s'=\downarrow \\ r'=\downarrow}}$$

$$= \sum_{s',r'} |A_{s',r'}|^2$$

- Suppose that you cannot measure the *initial* spins either

$\Rightarrow$ Average over initial spins

$$\overline{|A|^2} = \frac{1}{2}\frac{1}{2}\sum_{s,r} |A_{s,r}|^2 \quad \text{Average over initial spins}$$

$$P_{\text{total}} = P_{\substack{\text{inclusive on final states} \\ \text{averaged over initial states}}} = P_{\text{unpolarised}}$$

$$= \frac{1}{4}\sum_{\substack{s,r \\ s',r'}} |A_{s,r,s',r'}|^2 \quad (p_1, p_2, p_3, p_4)$$

Recall:

$$\boxed{\sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = (\not{p} - m\mathbb{1})_{\alpha\beta}}$$

$$\boxed{\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = (\not{p} + m\mathbb{1})_{\alpha\beta}}$$

$$\sum_{\substack{s,s'\\r,r'}} [\bar{v}^{r'}(p_4)\gamma_\mu u^{s'}(p_3)][\bar{u}^s(p_1)\gamma^\mu v^r(p_2)][\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)][\bar{v}^r(p_2)\gamma^\nu u^s(p_1)]$$

$$= \underbrace{\sum_{s'r'}\left(\bar{v}^{r'}(p_4)\gamma_\mu u^{s'}(p_3)\right)\left(\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right)}_{A_{\mu\nu}} \times$$

$$\times \underbrace{\sum_{r,r'}\left(\bar{u}^s(p_1)\gamma^\mu v^r(p_2)\right)\left(\bar{v}^r(p_2)\gamma^\nu u^s(p_1)\right)}_{\tilde{A}^{\mu\nu}}$$

$$A_{\mu\nu} = \sum_{s'r'}\sum_{\alpha,\beta,\delta,\epsilon}\left(\bar{v}_\alpha^{r'}(p_4)(\gamma_\mu)_{\alpha\beta} u^{s'}_\beta(p_3)\right)\left(\bar{u}^{s'}_\delta(p_3)(\gamma_\nu)_{\delta\epsilon} v^{r'}_\epsilon(p_4)\right) =$$

$$= \sum_{\alpha,\beta,\delta,\epsilon}\boxed{\sum_{s'} u^{s'}_\beta(p_3)\bar{u}^{s'}_\delta(p_3)}\boxed{\sum_{r'} v^{r'}_\epsilon(p_4)\bar{v}^{r'}_\alpha(p_4)}(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\delta\epsilon}$$

$$= \sum_{\alpha\beta\delta\epsilon}(\not{p}_3 + m_\mu\mathbb{1})_{\beta\delta}(\not{p}_4 - m_\mu\mathbb{1})_{\epsilon\alpha}(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\delta\epsilon}$$

$$= \underbrace{\sum_{\alpha\beta\delta\epsilon}{}' (\cancel{P}_4 - m_\mu \mathbb{1})_{\epsilon\alpha} (\gamma_\mu)_{\alpha\beta} (\cancel{P}_3 + m_\mu \mathbb{1})_{\beta\delta} (\gamma_\nu)_{\delta\epsilon}}$$

$$= \sum_{\epsilon}{}' \left\{ (\cancel{P}_4 - m_\mu \mathbb{1}) \gamma_\mu (\cancel{P}_3 + m_\mu \mathbb{1}) \gamma_\nu \right\}_{\epsilon\epsilon}$$

$$= \mathrm{Tr}\left[(\cancel{P}_4 - m_\mu \mathbb{1}) \gamma_\mu (\cancel{P}_3 + m_\mu \mathbb{1}) \gamma_\nu \right]$$

$$\equiv A_{\mu\nu}$$

$$\tilde{A}^{\mu\nu}(P_1, P_2) = \mathrm{Tr}\left[(\cancel{P}_2 - m_e \mathbb{1}) \gamma^\mu (\cancel{P}_1 + m_e \mathbb{1}) \gamma^\nu \right]$$

$$\mathrm{Tr}\left[(\cancel{P}_2 - m_e \mathbb{1}) \gamma^\mu (\cancel{P}_1 + m_e \mathbb{1}) \gamma^\nu \right] = \text{expand}$$

contains $\quad \mathrm{Tr}_{\alpha\beta}\, \gamma^\mu \gamma^\nu = \mathrm{Tr}\, \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \mathrm{Tr}_{\alpha\beta}\, \mathbb{1}\, \eta^{\mu\nu}$ 4

$$\mathrm{Tr}(\gamma^\mu P_\rho \gamma^\rho \gamma^\nu) = P_\rho \mathrm{Tr}(\gamma^\mu \gamma^\rho \gamma^\nu) = 0$$

$$\mathrm{Tr}(P_{2\sigma} \gamma^\sigma \gamma^\mu \overbrace{P_{1\rho} \gamma^\rho}^{\cancel{P}} \gamma^\nu) = P_{2\sigma} P_{1\rho} \mathrm{Tr}(\gamma^\sigma \gamma^\mu \gamma^\rho \gamma^\nu)$$

$$\mathrm{Tr}\,\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu = 4\left(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\sigma\rho}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\rho}\right)$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$$

$$\mathrm{Tr}\left[(\not{p}_2 - m_e \mathbb{1})\gamma^\mu(\not{p}_1 + m_e\mathbb{1})\gamma^\nu\right] =$$

$$= -m_e^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu] + p_{2\sigma}p_{1\rho}\,\mathrm{Tr}[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu]$$

$$= -4m_e^2\eta^{\mu\nu} + p_{2\sigma}p_{1\rho}\left(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\sigma\rho}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\rho}\right)$$

$$= -4\eta^{\mu\nu}m_e^2 + \left(p_2^\mu p_1^\nu - (p_2\cdot p_1)\eta^{\mu\nu} + p_2^\nu p_1^\mu\right)$$

$$= 4\left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_2\cdot p_1)\eta^{\mu\nu} - \eta^{\mu\nu}m_e^2\right]$$

$$\equiv \tilde{A}^{\mu\nu}$$

$$A_{\mu\nu} = 4\left[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - (p_3\cdot p_4)\eta_{\mu\nu} - \eta_{\mu\nu}m_\mu^2\right]$$

$$\frac{1}{4}\sum_{\text{polarisations}}' |A|^2 = \frac{e^4}{4s^2}\tilde{A}^{\mu\nu}A_{\mu\nu} =$$

$$= \frac{e^4}{\cancel{4}s^2}\cancel{4}\left[p_2^{\mu}p_1^{\nu} + p_2^{\nu}p_1^{\mu} - (p_2\cdot p_1)\eta^{\mu\nu} - \eta^{\mu\nu}m_e^2\right]$$
$$4\left[p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - (p_3\cdot p_4)\eta_{\mu\nu} - \eta_{\mu\nu}m_\mu^2\right] =$$

$$\boxed{= 8\frac{e^4}{s^2}\left[p_{14}p_{23} + p_{13}p_{24} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_\mu^2 m_e^2\right]}$$

where $p_{iJ} = p_i\cdot p_J$ (e.g. $p_{14} = p_1\cdot p_4$ ...)

$\equiv p_i^{\mu}p_{J\mu}$

<u>Introduce</u> Mandelstam variables:

- $s = (p_1+p_2)^2 \equiv (p_1+p_2)_\mu(p_1+p_2)^\mu = p_1^2 + p_2^2 + 2p_1\cdot p_2 = 2m_e^2 + 2p_{12}$

$$\Rightarrow \quad p_{12} = \frac{1}{2}s - m_e^2$$

Also

$p_1 + p_2 = p_3 + p_4 \Rightarrow s = (p_3+p_4)^2 = 2m_\mu^2 + 2p_{34}$

$$p_{34} = \frac{1}{2}s - m_\mu^2$$

- $t = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_{13} = m_e^2 + m_\mu^2 - 2p_{13}$

- $u = (p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_{14} = m_e^2 + m_\mu^2 - 2p_{14}$

$\Rightarrow$ Express P_{ij} in terms of s, t, u, m_e^2, m_μ^2

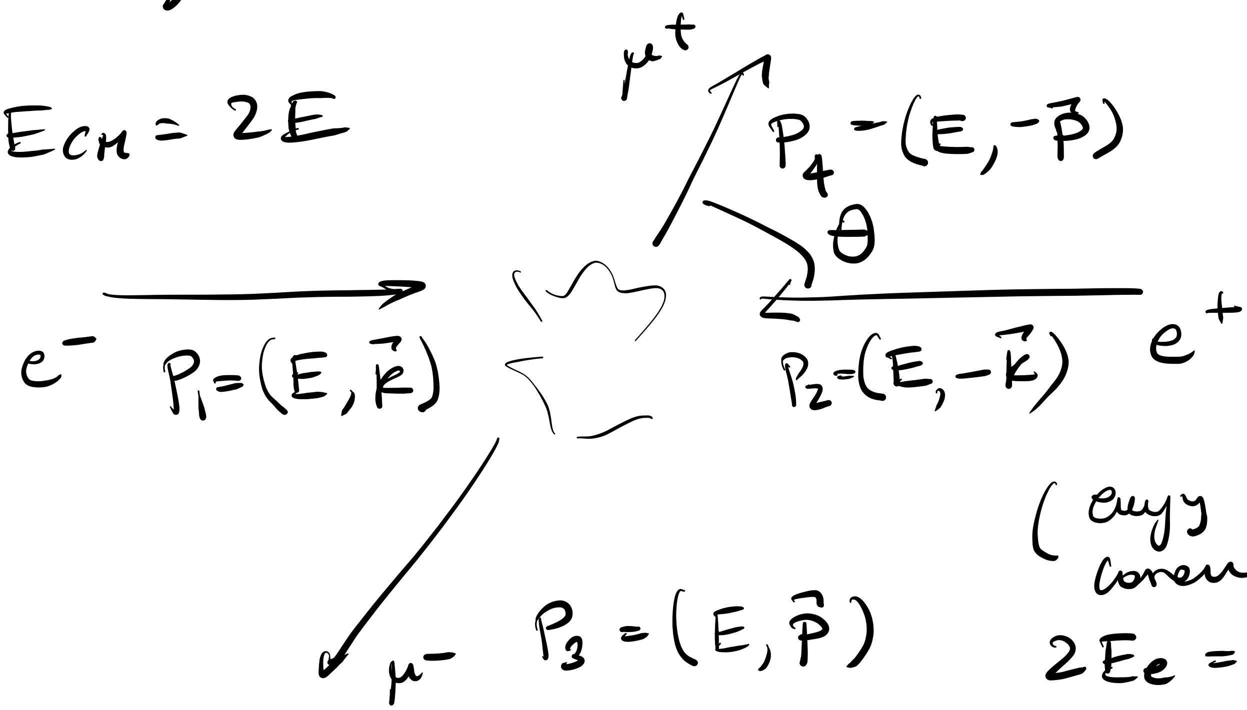
$P_{23} = \,?\quad u = (P_1 - P_4)^2 = (P_2 - P_3)^2 = m_e^2 + m_\mu^2 - 2P_{23}$

$$\Rightarrow \boxed{\frac{1}{4}\sum_{\text{spins}}|A|^2 = \frac{2e^4}{s^2}\Big[u^2 + t^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\Big]}$$

exercise

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2}\frac{1}{E_{CM}^2}\frac{|\vec{P}_f|}{|\vec{P}_i|}\left(\frac{1}{4}\sum|A|^2\right)$$

$E_{CM} = 2E$



(energy is conserved so

$2E_e = 2E_\mu$

$$s = (p_1+p_2)^2 = \big(\underbrace{(2E,\vec{0})}_{p_1+p_2}\big)^2 = 4E^2 = (E_{CM})^2$$

$$t = (p_1-p_3)^2 = \big((0,\vec{k}-\vec{p})\big)^2 = m_e^2 + m_\mu^2 - 2p_1\cdot p_3$$

$$p_1\cdot p_3 = (E,\vec{k})\cdot(E,+\vec{p}) = E^2 - \vec{k}\cdot\vec{p}$$

$$\Rightarrow t = m_e^2 + m_\mu^2 - 2(E^2 - \vec{k}\cdot\vec{p}) \quad \left(= (\vec{k}-\vec{p})^2\right)$$

$$u = m_e^2 + m_\mu^2 - 2(E^2 + \vec{k}\cdot\vec{p})$$

insert s, t, u into cross section

$$\alpha = e^2/4\pi$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{16E^6}\frac{|\vec{p}|}{|\vec{k}|}\left[E^4 + (\vec{k}\cdot\vec{p})^2 + E^2(m_e^2+m_\mu^2)\right]}$$

depends on E, θ

Relative Angle between e, μ

$$(\vec{k}\cdot\vec{p})^2 = \underbrace{(E^2-m_e^2)}_{|\vec{k}|^2}\underbrace{(E^2-m_\mu^2)}_{|\vec{p}|^2}\underbrace{\cos^2\theta}_{\cos^2\theta}$$

suppose that $E \gg m_e, m_\mu$

$$(\vec{k}\cdot\vec{p})^2 \simeq E^4 \cos^2\theta \,, \quad |\vec{p}|/|\vec{k}| \simeq 1$$

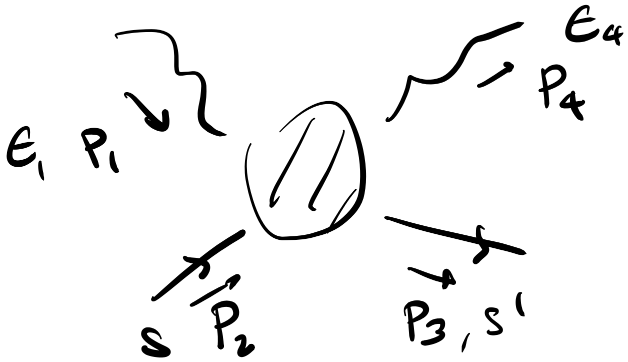
$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \simeq \frac{\alpha^2}{16E^6}\left(E^4 + E^4\cos^2\theta\right)$$

$$\boxed{= \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right)}$$

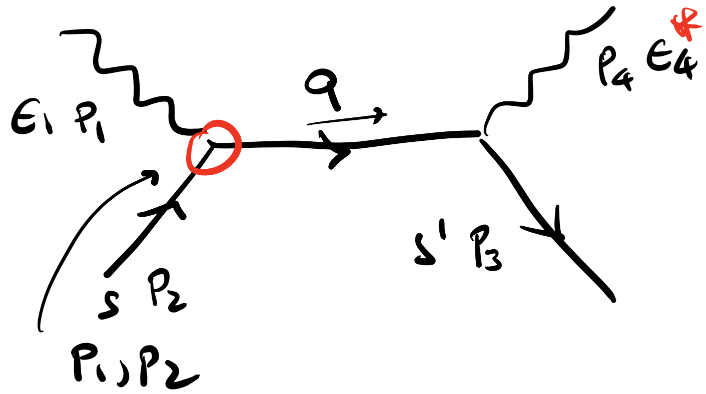
(unpolarised) ultra-relativistic $e^+e^- \to \mu^+\mu^-$ cross-section in CM frame

Compton Scattering

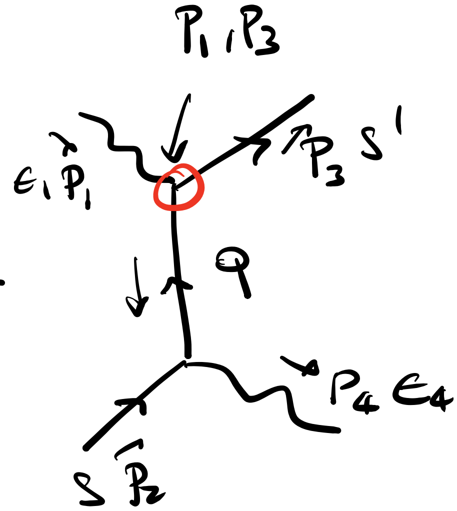
$$e^- \gamma \longrightarrow e^- \gamma$$



$A =$



$+$



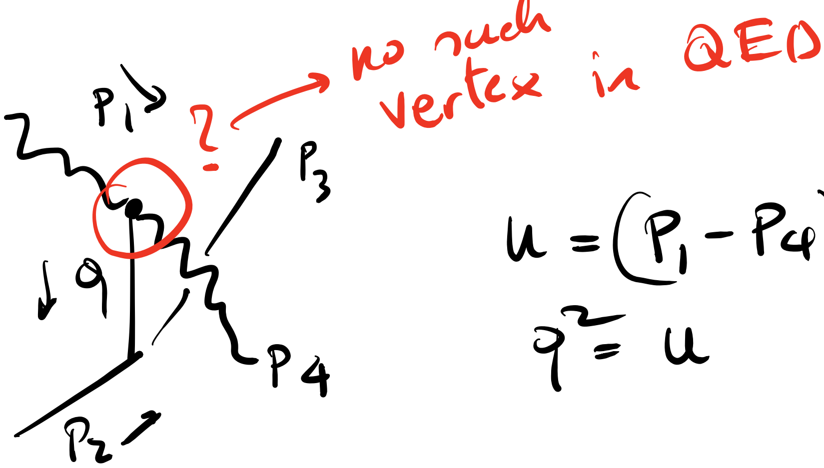
$$q = p_1 + p_2 \qquad q^2 = s$$

s- channel

$$q = p_1 - p_3 \qquad q^2 = t$$

t- channel

no u-channel here



$$u = (p_1 - p_4)^2$$

$$q^2 = u$$

ingoing particle : $\rightarrow u^s(p)$

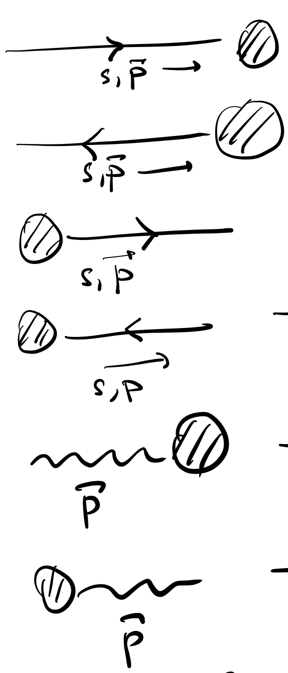
ingoing antiparticle $\rightarrow \bar{v}^s(\vec{p})$

outgoing particle $\rightarrow \bar{u}^s(\vec{p})$

outgoing antiparticle $\rightarrow v^s(\vec{p})$

incoming photon $\rightarrow \epsilon_\mu(p)$

outgoing photon $\rightarrow \epsilon^*_\mu(p)$

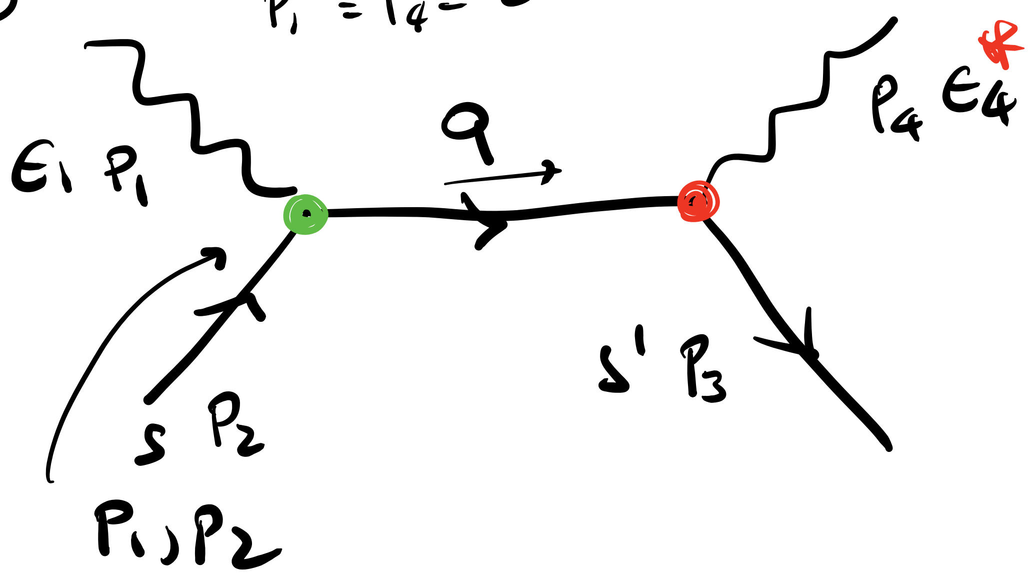


$$\vec{\epsilon}_4 \cdot \vec{p}_4 = 0$$
$$\epsilon_4^{\mu} \, p_{4\,\mu} = 0$$

$$p_1^2 = p_4^2 = 0$$

$$\epsilon_1^{\mu} \, p_{1\,\mu} = 0$$
$$\vec{\epsilon}_1 \cdot \vec{p}_1 = 0$$

$A_s =$



$=$

$$= \bar{u}^{s'}_{\alpha}(p_3)\left(-ie\gamma^{\mu}_{\alpha\beta}\right)\frac{\left(\overset{\cancel{p}_1 + \cancel{p}_2}{\cancel{q}} + m_e \mathbb{1}\right)_{\beta\delta}}{s - m_e^2 + i\epsilon}\left(-ie\gamma^{\nu}_{\delta\epsilon}\right)u^{s}_{\epsilon}(p_1)\; \epsilon_{1\nu}(p_1)\, \epsilon^*_{4\mu}(p_4)$$

$$A_t = -e^2 \epsilon_1^{\mu} \epsilon_4^{*\nu} \, \bar{u}(p_3)\gamma_\mu \frac{\left[(\cancel{p}_1 - \cancel{p}_3) + m\right]}{t - m_e^2 + i\epsilon}\gamma_\nu u(p_2)$$

Unpolarized probability:

$$\frac{1}{4} \sum_{\text{spins}}{}' \sum_{\text{polarizations}}{}' \left| A_s + A_t \right|^2 =$$

$$= \frac{1}{4} \sum_{\text{spins}}{}' \sum_{\substack{\text{polarizations}\\ \text{physical}}}{}' \left(A_s^\dagger A_s + A_t^\dagger A_t + A_s^\dagger A_t + A_t^\dagger A_s \right)$$

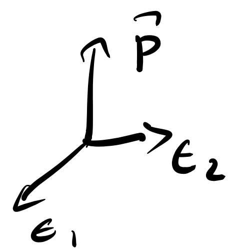
Polarization sums:

" $\sum_n{}' |n\rangle\langle n| = \mathbb{1}$ "

$$\sum{}' \epsilon_p^\mu(\vec{p})\, \epsilon_p^\nu(\vec{p}) = -\eta^{\mu\nu}$$

Complete basis of vectors

$$p_\mu, \ \bar{p}_\mu, \ \epsilon_1, \ \epsilon_2$$



$$\begin{cases} p_\mu = (|p|, \vec{p}) \\ \bar{p}_\mu = (|p|, -\vec{p}) \\ \boxed{\epsilon_1, \epsilon_2 = (0, \vec{\epsilon}_{1,2})} \end{cases}$$

physical

$$-\eta^{\mu\nu} = \underbrace{\sum^{1} \epsilon_p^{\mu}(\vec{p})\epsilon_p^{\nu}(\vec{p})}_{\substack{\text{Complete} \\ \text{basis of} \\ \text{vectors}}} = \underbrace{\sum_{i=1}^{2}}_{\perp \text{ polarisations}} \epsilon_i^{\mu}\epsilon_i^{\nu} + \underbrace{\sum^{1} \epsilon_L^{\mu}\epsilon_L^{\nu}}_{\parallel \text{ polarisation}}$$

$$\epsilon^{\mu\,\parallel} = \{(|\vec{p}|, \vec{p}), (|\vec{p}|, -\vec{p})\}$$ longitudinal

$$\epsilon_i^{\mu} = \{(0, \vec{\epsilon}_1), (0, \vec{\epsilon}_2)\}$$

$$\sum_{\text{longitudinal}}^{1} \epsilon_L^{\mu}(p)\epsilon_L^{\nu}(p) = \frac{p^{\mu}\bar{p}^{\nu} + \bar{p}^{\mu}p^{\nu}}{2E^2}$$

$$\Rightarrow \boxed{\sum_{\perp\, i=1}^{2} \epsilon_i^{\mu}(p)\epsilon_i^{\nu}(p) = -\eta^{\mu\nu} + \frac{p^{\mu}\bar{p}^{\nu} + \bar{p}^{\mu}p^{\nu}}{2E^2}}$$

this never contributes !

$$A_s = \bar{u}_{\alpha}^{s'}(p_3)(-ie\gamma^{\mu}_{\alpha\beta})\frac{(\overset{\not p_1 + \not p_2}{\not q} + m_e \mathbb{1})_{\beta\delta}}{s - m_e^2 + i\epsilon}(-ie\gamma^{\nu}_{\delta\epsilon})u^{s}_{\epsilon}(p_1)\,\epsilon_{1\nu}(p_1)\,\epsilon^{*}_{4\mu}(p_4)$$

$$= \epsilon_{1\nu}(p_1)\,A_s \;\checkmark$$

Fact:

$$P_{1\mu} A_s^{\mu} = 0$$

$$\bar{P}_{1\mu} A_s^{\mu} = 0$$

Effectively:

$$\text{"} \sum_{\perp \text{ polarizations}}' \epsilon_i^{\mu}(p)\, \epsilon_i^{\nu}(p) = -\eta^{\mu\nu} \text{"}$$

(true when contracted into an amplitude)

$$\frac{1}{4}\sum_{\substack{\text{spins}\\ \text{pol}}}' |\mathcal{A}|^2 = \text{------}$$

spin sum rules + polarization sum rule

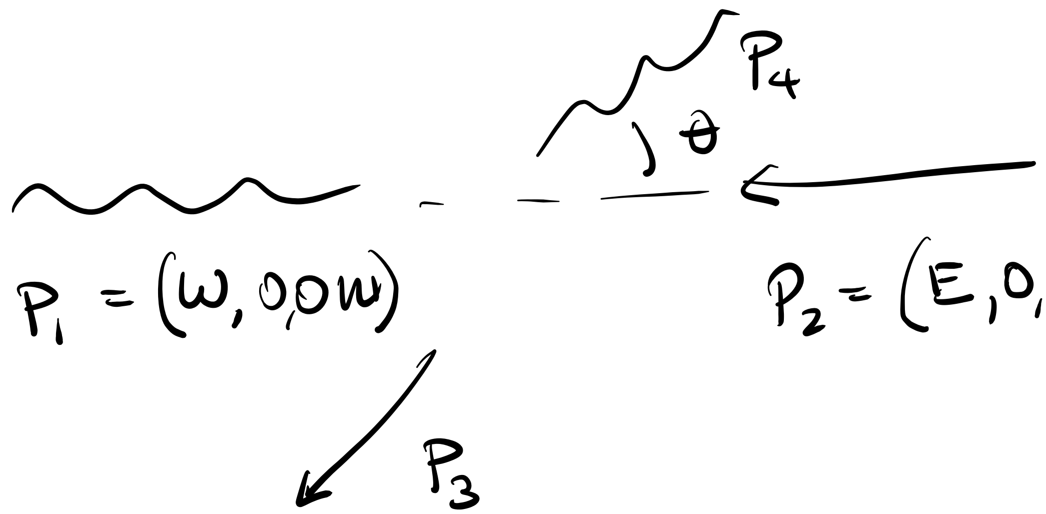
$$= e^4 \,\mathrm{Tr}\left[(\not{p}_3+m)\left(\frac{\gamma^\nu(\not{p}_1+\not{p}_2+m)\gamma^\mu}{s-m^2}+\frac{\gamma^\mu(\not{p}_2-\not{p}_4+m)\gamma^\nu}{t-m^2}\right)\right.$$

$$\left.(\not{p}_2+m)\left(\frac{\gamma_\mu(\not{p}_1+\not{p}_2+m)\gamma_\nu}{s-m^2}+\frac{\gamma_\nu(\not{p}_2-\not{p}_4+m)\gamma_\mu}{t-m^2}\right)\right]$$

use trace identities ......

$$=2e^4\left[\frac{P_{24}}{P_{12}}+\frac{P_{12}}{P_{24}}+2m^2\left(\frac{1}{P_{12}}-\frac{1}{P_{24}}\right)+m^4\left(\frac{1}{P_{12}}-\frac{1}{P_{24}}\right)^2\right]$$

Kinematics in CM frame



$E = \sqrt{\omega^2 + m^2}$

$P_1 = (\omega, 0, 0, \omega)$ $\qquad$ $P_2 = (E, 0, 0, -\omega)$

$$P_4 = (\omega, \omega \sin\theta, 0, \omega\cos\theta)$$
$$P_3 = (E, -\omega\sin\theta, 0, -\omega\cos\theta)$$

$$\Rightarrow \quad P_{12} = \omega E + \omega^2$$
$$P_{24} = \omega(E + \omega\cos\theta)$$

$\omega \gg m$ (photon has very high energy)

$$\frac{1}{4}\sum|A|^2 \simeq 4e^4\left[\frac{1+\cos\theta}{4} + \frac{1}{\underbrace{\frac{m^2}{4\omega^2}}_{\text{small}} + 1 + \cos\theta}\right]$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{64\pi^2(2\omega)^2} \times \frac{1}{4}\sum|A|^2 =$$

$$\boxed{= \frac{\pi\alpha^2}{2\omega^2}\left[\frac{1+\cos\theta}{4} + \frac{1}{\frac{m^2}{2\omega^2}+1+\cos\theta}\right]}$$

Compton cross section in the CM frame in the high-energy limit