Spontaneous Symmetry Breaking

• Vacua of FT

take a scalar field theory (classical)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi)$$

(ex: in KG theory $V(\phi) = \frac{1}{2}m^2\phi^2$ (free)

in interacting theories

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \dots$$)

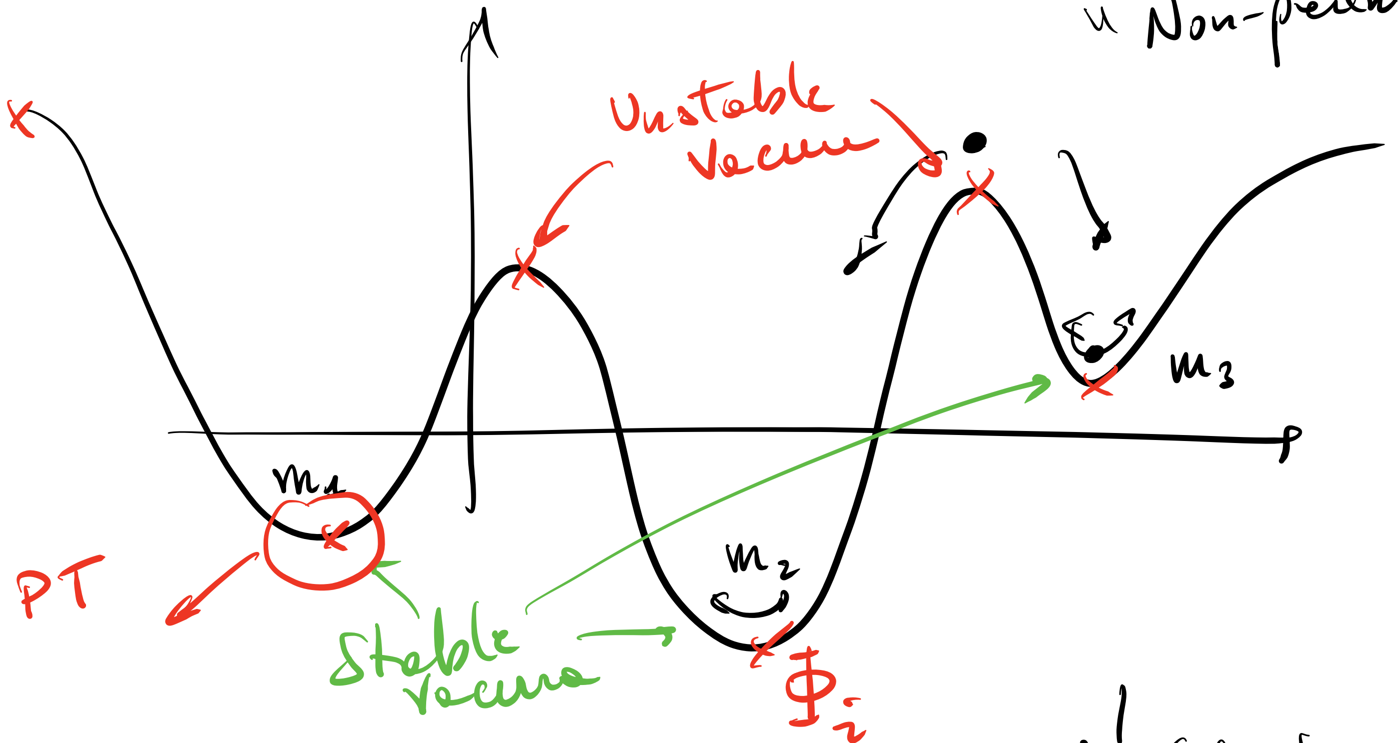
$$\Box \Phi + \frac{dV}{d\Phi} = 0 \quad \text{(in general non-linear)}$$

suppose that $\exists\ \Phi_0$ such $V'(\Phi_0) = 0$

$\rightarrow$ $\phi(x) = \Phi_0\ \forall x$ is a solution of the EL Equation

It is a solution with zero kinetic energy, also it preserves Poincaré invariant. It is called a (classical) vacuum solution

" Non-perturbative "



Each Φ_i satisfies $V'(\Phi_i) = 0$ and corresponds to a vacuum solution

- Consider a small perturbation around Φ_i

$$\Phi(x) = \underset{x^\mu\text{-independent}}{\Phi_i} + \delta\Phi(x^\mu)$$

$$\Box\Phi(x) + \frac{dV}{d\Phi} = 0 \qquad \text{Exact}$$

Linearize in $\delta\phi$ and Φ_i

$$\Box\phi(x) = \cancel{\Box\Phi_i} + \Box\delta\Phi(x^\mu)$$

$$\frac{dV}{d\phi} = \cancel{\frac{dV}{d\phi}\Big|_{\phi_i}} + \frac{d^2V}{d\Phi^2}\Big|_{\phi_i}\delta\Phi$$

$$\Box\, \delta\Phi(x^\mu) + \left.\frac{d^2V}{d\Phi^2}\right|_{\Phi_i} \delta\Phi = 0$$

constant

$= m^2(\Phi_i)$

linear eq. in $\delta\Phi$

$$\left(\Box + m^2(\Phi_i)\right)\delta\Phi = 0$$

Free KG eq with mass depending on which vacuum state we chose

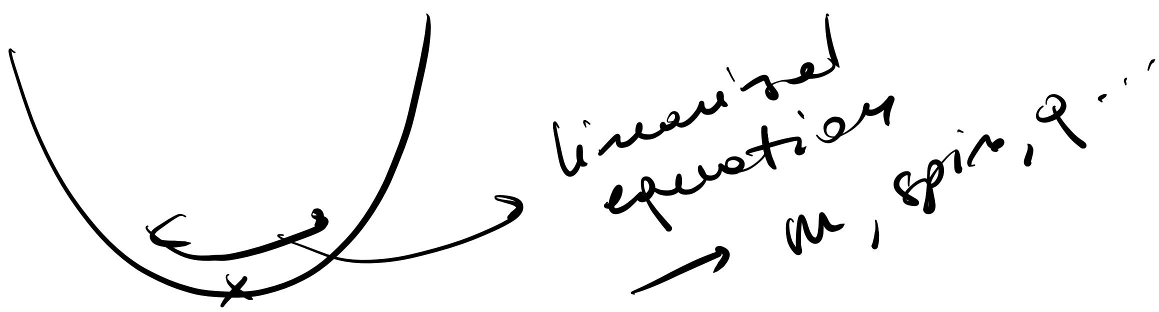
- if $m^2(\Phi_i) < 0$ (Φ_i is a maximum for V)
  $\Rightarrow$ unstable perturbations
- if $m^2 > 0$ then
  - vacuum is stable
  - Can build a Quantum theory for the fluctuations $\delta\Phi$

$$\widehat{\delta\Phi}(x) = \int \frac{d^3p}{(2\pi)^3}\left[\hat{a}\, e^{-ip\cdot x} + \hat{a}^\dagger e^{ip\cdot x}\right]$$

$$p^\mu = (\omega_p, \vec{p}) \qquad \omega_p = \sqrt{m_i^2 + \vec{p}^2}$$

$\Rightarrow$ Different "classical" vacua support different QFTs

<u>Spectrum</u> associated to a certain vacuum = set of all field excitations (masses, charges, spins) associated to that vacuum $\Leftrightarrow$ properties of <u>free</u> particles associated to fluctuations around that vacuum



• Interactions :

$$\Box \phi + \frac{dV}{d\phi} = 0 \qquad \Phi = \phi_i + \delta\phi$$

$$\rightarrow \quad 0 = \Box \delta\phi + \underset{\downarrow}{m^2}\, \delta\phi + \underbrace{\frac{1}{2}\frac{d^3V}{d\phi^3}\Big|_{\phi_i}}_{g} (\delta\phi)^2$$

$$m^2 \to V''(\phi_i)$$

expand to higher order in ϕ

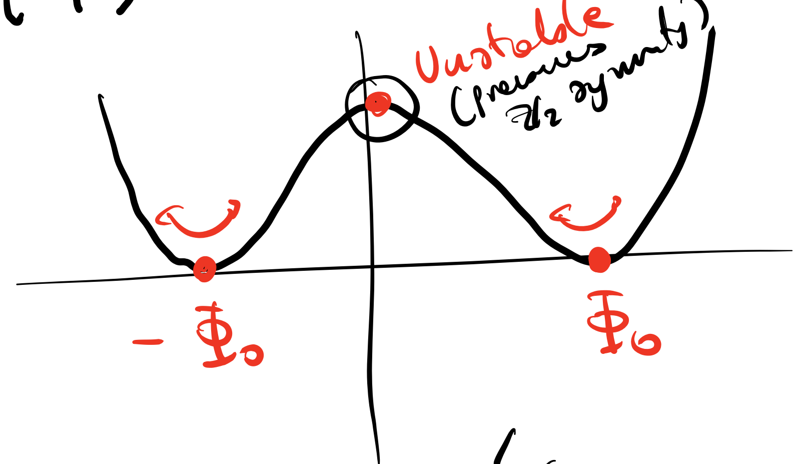
$\rightarrow$ KG theory with a <u>cubic</u> interaction

$$\left(\mathcal{L} = \frac{1}{2}\partial_\mu(\delta\phi)\partial^\mu(\delta\phi) - \frac{1}{2}m^2\delta\phi^2 - \frac{g}{3}(\delta\phi)^3 \right)$$

(For theories with only <u>relevant</u> or <u>marginal</u> couplings $\Leftrightarrow$ V a quartic polynomial $\Leftrightarrow$ only up to 4^{th} derivatives of V are $\neq 0$)

- Suppose $V(\phi)$ has an internal symmetry: e.g. $\Phi \longrightarrow -\Phi$ ($\mathbb{Z}_2$)

$$V(-\phi) = V(\phi) \quad \text{(even under } \mathbb{Z}_2\text{)}$$



$$V'(-\Phi_0) = V'(\Phi_0) \quad \forall\ \phi_0 \neq 0$$

$\Rightarrow$ if Φ_0 is a vacuum $\Rightarrow$ so is $-\Phi_0$

2 QFTs: $\Phi = \Phi_0 + \hat{\varphi}^{(+)}(x)$ $\rightarrow$ $\delta\phi$ around ϕ_0

$\phi = -\Phi_0 + \hat{\varphi}^{(-)}(x)$ $\rightarrow$ $\delta\phi$ around $-\phi_0$

$\Rightarrow$ Two (equivalent) QFTs

$$|0^{(+)}\rangle \qquad \begin{cases} \langle 0^{(+)}|\phi|0^{(+)}\rangle = \phi_0 \\ \langle 0^{(-)}|\phi|0^{(-)}\rangle = -\phi_0 \end{cases}$$
$$|0^{(-)}\rangle$$

in particular $|0^{(-)}\rangle$ is <u>not</u> a state in the QFT with vacuum $|0^{(+)}\rangle$

<u>however</u>:

$$\mathbb{Z}_2 : |0^{(+)}\rangle \longrightarrow |0^{(-)}\rangle$$
$$(\phi_0 \longrightarrow -\phi_0)$$

<u>Neither</u> <u>vacuum</u> <u>is</u> <u>invariant</u> <u>under $\mathbb{Z}_2$</u>

" $\mathbb{Z}_2$ symmetry is spontaneously broken"

<u>broken</u>: vacuum is not invariant

<u>Spontaneously</u>: Lagrangian <u>is</u> invariant

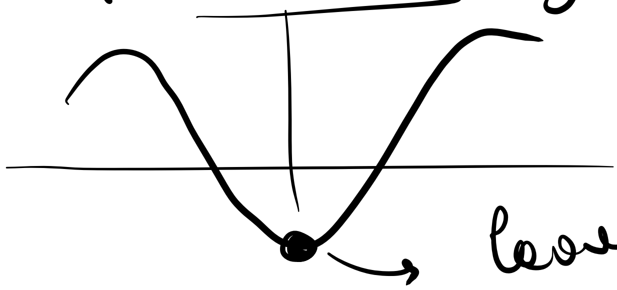
On contrary, suppose you had chose

$\Phi_0 = 0$ $\qquad \mathbb{Z}_2 : \phi \to -\phi$

invariant $\qquad \phi = 0 \to \phi = 0$

• Example of unbroken symmetry:

V



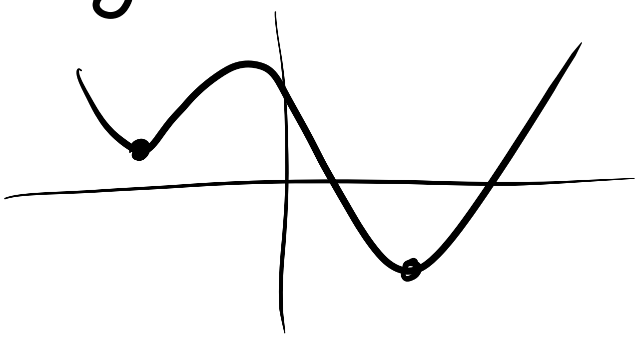
leaves $\mathbb{Z}_2$ unbroken

$$V(-\phi) = V(\phi) \qquad \phi_0 = 0 \xrightarrow[\mathbb{Z}_2]{} \phi_0 = 0$$

• Explicitely Broken symmetry:

$\mathcal{L}$ is not invariant $V(-\phi) \neq V(\phi)$

(symmetry is not there in the first place)



$\cancel{\phi \to -\phi}$

broken by the form of the potential

Continuos Global symmetry

take a complex scalar field Φ and:

$$\mathcal{L} = (\partial^\mu \phi)^* (\partial_\mu \phi) - V(\phi^* \phi)$$

invariant under a (global) $U(1)$:

$$\phi \to e^{i\alpha} \phi \qquad \mathcal{L} \to \mathcal{L}$$
$$\phi^* \to e^{-i\alpha} \phi^* \qquad \alpha \in \mathbb{R}$$

$\Longleftrightarrow$ associated Nöther current

$$J_\mu(x) \qquad \partial^\mu J_\mu$$

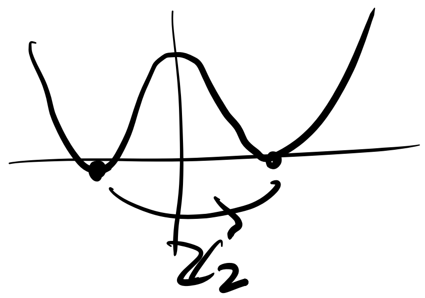
$$Q = \int d^3x \, J_0 \quad \text{is conserved}$$

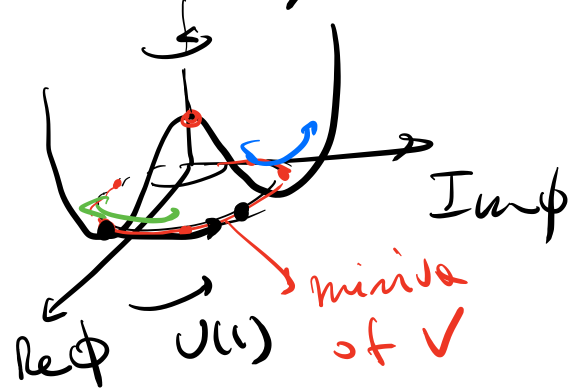
Goldstone Theorem

if a continuos symmetry is spontaneously broken $\Longrightarrow$ $\exists$ a massless particle in the QFT spectrum

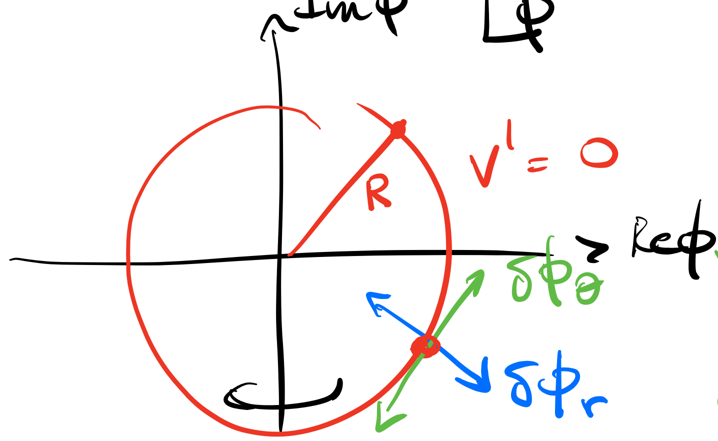
- More specifically: there is one massless particle for each broken symmetry generator.

$$L = (\partial_\mu \phi)^*(\partial^\mu \phi) - V(\phi^* \phi)$$





$L\phi$



$\delta\phi_\theta$ can be turned on with as little energy as you want

$\Rightarrow \widehat{\delta\phi_\theta}$ corresponds to a massless particle $\quad \Box\,\delta\phi_\theta = 0$

Explict example:

- $V = m^2 |\phi|^2 + \frac{\lambda}{4}(|\phi|^2)^2$

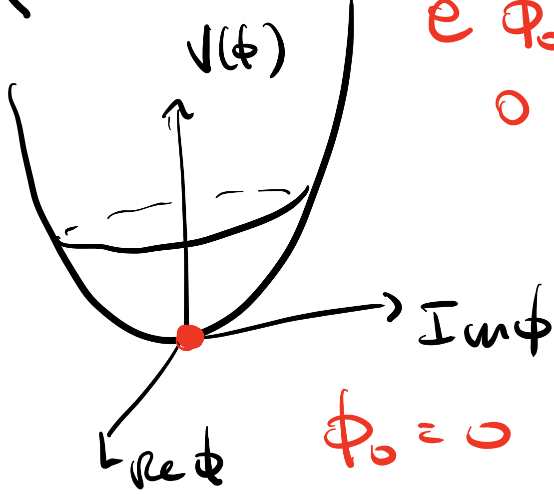
U(1) is unbroken

$e^{i\alpha}\phi_0 = \phi_0$

$0 = 0$

suppose $m^2 > 0$

$V' = 0$ has only solution $\phi = 0$



Spectrum: $\Phi = \phi_0 + \delta\phi = \delta\phi$

$$\Box\,\delta\phi + \underbrace{V''(0)}_{m^2}\,\delta\phi + \frac{1}{2}\underbrace{V'''(0)}_{0}(\delta\phi)^2 + \frac{1}{3!}\underbrace{V''''(0)}_{\lambda}\,|\delta\phi|^2\,\delta\phi^*$$

$$\Rightarrow \begin{cases} \dfrac{\delta L}{\delta\phi^*} \to & (\Box + m^2)\,\delta\phi + \lambda|\delta\phi|^2\,\delta\phi^* = 0 \\[2ex] \dfrac{\delta L}{\delta\phi} \to & (\Box + m^2)\,\delta\phi^* + \lambda|\delta\phi|^2\,\delta\phi = 0 \end{cases}$$

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$$

$$\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$

$$\phi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$$

φ_1, φ_2 Real scalars

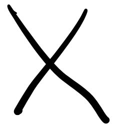
$$\mathcal{L} = \boxed{\frac{1}{2}(\partial_\mu\varphi_1)(\partial^\mu\varphi_1) + \frac{1}{2}(\partial_\mu\varphi_2)(\partial^\mu\varphi_2) - \frac{m^2}{2}\varphi_1^2 - \frac{m^2}{2}\varphi_2^2} \boxed{- \frac{\lambda}{4}\frac{1}{4}(\varphi_1^2 + \varphi_2^2)^2}$$

↳ spectrum: 2 spin-0 particles with the same mass m

——— 1

- - - - 2

Interactions:

 $\lambda \frac{4!}{16}$  $\lambda \frac{4!}{16}$  $\lambda \frac{4!}{8}$

$$\square = \frac{\lambda}{16}\left(\varphi_1^4 + \varphi_2^4 + 2\varphi_1^2\varphi_2^2\right) \cancel{\boxed{+ \alpha\, \varphi_1^2 \varphi_2^2}}$$

would break U(1) symmetry explicitly

- $$V = m^2 (\phi^*\phi) + \frac{\lambda}{4}(\phi^*\phi)^2$$

$$m^2 = -\mu^2 < 0 \qquad (\mu \text{ real})$$

Vacua:

$$\begin{cases} 0 = \dfrac{\partial V}{\partial \phi} = -\mu^2 \phi^* + \dfrac{\lambda}{2}\phi^*(\phi^*\phi) \\ 0 = \dfrac{\partial V}{\partial \phi^*} = -\mu^2 \phi + \dfrac{\lambda}{2}\phi(\phi^*\phi) \end{cases}$$

- $\phi_0 = 0$ unstable

- $\phi \neq 0$

(1) $\Rightarrow -\mu^2 + \frac{\lambda}{2}(\phi^*\phi) = 0$

(2) $\rightarrow -\mu^2 + \frac{\lambda}{2}(\phi^*\phi) = 0$

$\Rightarrow$ $$\boxed{\phi^*\phi = \frac{2\mu^2}{\lambda}}$$

Absolute minima of $V(\phi)$

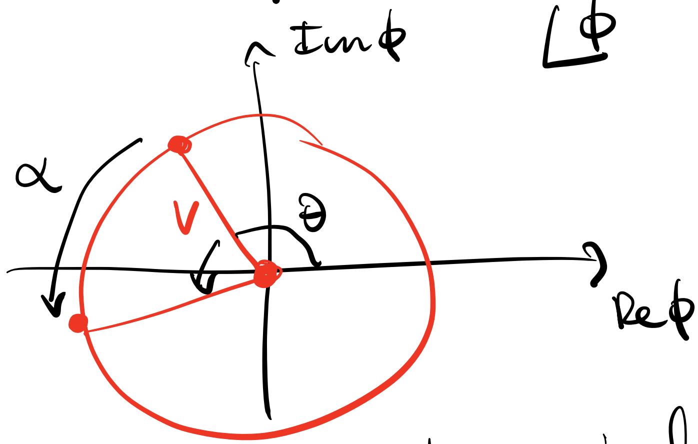
General solution

$=0$

$$\phi = v e^{i\theta} \qquad v = \sqrt{\frac{2\mu^2}{\lambda}}$$

$\angle\phi$

$\theta \in [0, 2\pi]$ arbitrary



each point of the circle is a vacuum state (stable)

$$U(1): \ \phi \to e^{i\alpha}\phi$$

so $$v e^{i\theta} \longrightarrow v e^{i(\theta+\alpha)} \neq v e^{i\theta}$$

$\Rightarrow$ None of the vacua is invariant

$\Rightarrow$ U(1) spontaneously broken

Spectrum

Choose vacuum $\phi = v$ ($\theta = 0$)

$$\Phi(x) = \Big(\underbrace{v}_{\text{Background}} + \frac{\rho(x)}{\sqrt{2}}\Big) e^{i\pi(x)/f_\pi}$$

$$f_\pi = \sqrt{2}\, v$$

$\rho(x) \in \mathbb{R}$ radial fluctuation

$\pi(x) \in \mathbb{R}$ phase fluctuation (angular)

(We could have written:

$$\phi(x) = v + \frac{(\varphi_1 + i\varphi_2)}{\sqrt{2}}$$)

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2$$

where $\Phi = (v + \rho/\sqrt{2}) e^{i\pi/f_\pi}$

at $\underline{\text{quadratic}}$ level in ρ, π

$$\mathcal{L} = \partial_\mu \left[\left(v + \frac{\rho}{\sqrt{2}}\right) e^{-i\pi/f_\pi} \right] \partial^\mu \left[\left(v + \frac{\rho}{\sqrt{2}}\right) e^{i\pi/f_\pi} \right]$$

$$- m^2 \left(v + \frac{\rho}{\sqrt{2}}\right)^2 - \frac{\lambda}{4} \left(v + \frac{\rho}{\sqrt{2}}\right)^4$$

$\partial_\mu v = 0$

$$= \left(\frac{1}{\sqrt{2}} (\partial_\mu \rho) \cancel{e^{-i\pi/f}} + \left(v + \frac{\rho}{\sqrt{2}}\right)\left(\frac{-i}{f}(\partial_\mu \pi)\right) \cancel{e^{-i\pi/f}} \right)$$

$$\left(\frac{1}{\sqrt{2}} (\partial^\mu \rho) \cancel{e^{+i\pi/f}} + \left(v + \frac{\rho}{\sqrt{2}}\right)\left(\frac{i}{f}(\partial^\mu \pi)\right) \cancel{e^{+i\pi/f}} \right)$$

$$- m^2 \left(v + \frac{\rho}{\sqrt{2}}\right)^2 - \frac{\lambda}{4} \left(v + \frac{\rho}{\sqrt{2}}\right)^4 =$$

$\cancel{\rho^2}$ $\qquad$ $\rho^2 (\partial_\mu \pi)(\partial^\mu \pi)$ $\qquad$ cubic and quartic interactions for ρ

$$= \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{1}{f_\pi^2}\left(v+\frac{\rho}{\sqrt{2}}\right)^2 \underbrace{(\partial_\mu\pi)(\partial^\mu\pi)}_{2\text{ field}}$$

$$- m^2\left(v+\rho/\sqrt{2}\right)^2 - \frac{\lambda}{4}\left(v+\rho/\sqrt{2}\right)^4$$

At quadratic order

$$\simeq \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{v^2}{f_\pi^2}(\partial_\mu\pi)(\partial^\mu\pi)$$

$$+ \mu^2\left(v^2 + \frac{2v\rho}{\sqrt{2}} + \frac{\rho^2}{2}\right)$$

$$- \frac{\lambda}{4}\left(v^4 + \frac{4v^3\rho}{\sqrt{2}} + \frac{6v^2\rho^2}{2} + \cancel{O(\rho^3)}\right)$$

$$= \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{1}{2}(\partial_\mu\pi)(\partial^\mu\pi)$$

$f_\pi \equiv \sqrt{2}\, v$ v $\overset{0}{\shortparallel}$

$$+ \left(v^2\mu^2 - \frac{\lambda}{4}v^4\right) + \cancel{\left(\frac{2\mu^2}{\sqrt{2}} - \frac{\lambda v^2}{\sqrt{2}}\right)}v\rho +$$

$$v = \sqrt{\frac{2\mu^2}{\lambda}}$$

$$+ \frac{1}{2}\underbrace{\left(\mu^2 - \frac{3}{2}v^2\lambda\right)}_{\substack{\mu^2 - 3\mu^2 \\ = -2\mu^2}}\rho^2$$

$$\mathcal{L}^{(2)}(\rho,\pi) = \boxed{\frac{1}{2}(\partial_\mu \rho)(\partial^\mu \rho) - \frac{1}{2}(2\mu^2)\rho^2 + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) \cancel{- \frac{m_\pi^2}{2}\pi^2}}$$

Spectrum :

1 massive particle ρ , $m_\rho^2 = 2\mu^2$

1 massless particle π $m_\pi^2 = 0$

F.E. $\begin{cases} \Box \rho + m_\rho^2 \rho = 0 \\ \Box \pi = 0 \end{cases}$

Higgs Mechanism

(a.k.a. spontaneous breaking of gauge symmetries)

take a theory with a U(1) local symmetry, same potential as before:

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi^*\phi)$$

Abelian Higgs model

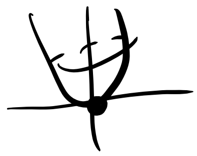
$$\phi(x) \to e^{ie\alpha(x)}\phi(x)$$

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$$

$$\boxed{V = -\mu^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2}$$

[$V = m^2\phi^*\phi + \dots$ $m^2>0$: spectrum : $\left\{\begin{array}{l}\cdot\ \text{charged massive } m^2 \text{ spin-0 particle } (\pm e\text{, same mass})\\ \cdot\ \text{massless spin-1 particle}\end{array}\right.$]

Vacuum: $\phi = 0$



Vacua (stable) $\dfrac{\partial V}{\partial\phi} = \dfrac{\partial V}{\partial\phi^*} = 0$

$\Rightarrow$ $\Phi = v e^{i\theta}$ $\quad v = \sqrt{\dfrac{2\mu^2}{\lambda}}$

$$\mathcal{L} = (D_\mu \phi)^*(D^\mu \phi) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - V(\phi^*\phi)$$

expand to quadratic order around

$\phi = v$

$$\Phi(x) = \left(v + \frac{h(x)}{\sqrt{2}} \right) \cancel{e^{i\frac{\pi(x)}{f_\pi}}} \qquad f_\pi = \frac{v}{\sqrt{2}}$$

Let's make a <u>local</u> U(1) transformation

$$\Phi(x) \longrightarrow \phi(x)\, e^{iq\alpha(x)}$$

where $\boxed{\alpha(x) = -\frac{1}{q}\frac{\pi(x)}{f_\pi}}$

Restrict to
$\rightarrow$ $\phi(x) = v + \frac{h(x)}{\sqrt{2}}$ $\quad h \in \mathbb{R}$

(Gauge-fixing) $\quad$ <u>A_μ is arbitrary</u>

(Unitary gauge)

A_μ has (a priori) 4 components

(actually $\underline{\underline{3}}$)

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) + \mu^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= (\partial_\mu + ieA_\mu)\left(v+\frac{h}{\sqrt{2}}\right)(\partial^\mu - ieA^\mu)\left(v+\frac{h}{\sqrt{2}}\right) - \frac{1}{2}(2\mu^2)h^2 + O(h^3) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \boxed{\frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(2\mu^2)h^2} + \boxed{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2v^2A_\mu A^\mu}$$ + higher order

Real massive scalar field (Higgs boson) $h(x)$ with mass $m_h^2 = 2\mu^2$,

neutral ($U(1): h \to h$)

Spin-1 Field, massive

$$\boxed{m_A = 2e^2v^2}$$

v = vacuum expectation value of Φ

- All particles are **massive** (even A_μ)
- Gauge invariance is **unbroken** at the level of $\mathcal{L}$ (however it is spontaneously broken by the VeV of Φ)

VeV = Vacuum expectation value

What even happened to $\pi(x)$?

AKA Proca Equations aka massive spin 1

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$

$$\boxed{0 = -\partial^\mu F_{\mu\nu} - m^2 A_\nu}$$

massive spin-1 F.E.

$$0 = \partial^\nu (\cancel{\partial^\mu F_{\mu\nu}} + m^2 A_\nu)$$

$$\partial^\nu \partial^\mu F_{\mu\nu} = 0$$

$$m^2 (\partial^\nu A_\nu) = 0$$

Lorentz condition is a ~~Field~~ Eq constraint!

$$\boxed{} = \partial^\mu \partial_\mu A_\nu - \partial_\nu \cancel{(\partial^\mu A_\mu)} + m^2 A_\nu = 0$$

$$\begin{cases} (\Box + m^2) A_\nu = 0 \\ \partial^\nu A_\nu = 0 \end{cases}$$

$\Rightarrow$ A_μ has $\cancel{4}$ 3 componts

$$\begin{cases} 2 \text{ transverse componts} & \vec{p} \cdot \vec{A}^\perp = 0 \\ 1 \text{ longitudinal compont} & A^{\parallel} \end{cases}$$

$$p^\mu A_\mu = 0$$

$$A_\mu = \sum_{s=1}^{\cancel{2}\,3} \int \frac{d^3p}{(2\pi)^3} \left(a^s \epsilon^s_\mu(p) e^{-ip\cdot x} + c.c. \right)$$

$p^\mu \epsilon_\mu = 0 \Rightarrow$ 3 vectors for each $\vec{p}$.

$\Pi(x)$ has become the extra polarisation of A_μ!

<u>Higgs Mechanism</u>: the gauge field acquires a (gauge-invariant) mass due to a scalar field vacuum expectation value which breaks the Gauge symmetry spontaneously