Non - Abelian Gauge theories

invariance under a non-abelian internal symmetry based on a group G

(think of $SU(2)$. e.g.: isospin

- start with a theory with global $SU(2)$ invariance

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{C}^2$$

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - V(\phi^\dagger \phi)$$

transform: $\Phi \longrightarrow U\phi \qquad U \in SU(2)$

$$\begin{cases} U^\dagger U = \mathbb{1} \\ \det U = 1 \end{cases} \qquad \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{U} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

Global symmetry: U is a constant matrix

$\mathcal{L} \longrightarrow \mathcal{L}$ is invariant

$$U = \exp\left[i \sum_{a=1}^{3} \theta_a \tau_a \right] \qquad \theta_a \in \mathbb{R}$$

$\tau_a = \sigma_a / 2$

Pauli 2x2 Matrices

$[\tau_a, \tau_b] = i\varepsilon_{abc} \tau_c$

τ_a : generators of infinitesimal transformation

$\theta_a \ll 1 \Rightarrow U \simeq \mathbb{1} + \omega \qquad \omega = \sum \theta_a \tau_a$

they form a Lie Algebra $\mathcal{A}$: $\begin{cases} \omega + \lambda\omega' \in \mathcal{A} \\ [\omega, \omega'] \in \mathcal{A} \\ [\omega,[\omega',\omega'']] + \text{cyclic} = 0 \end{cases}$

$$\delta\Phi = i\sum_{a=1}^{3}\theta_a \tau_a \Phi \Leftrightarrow \delta\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = i\begin{pmatrix}\theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & -\theta_3\end{pmatrix}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$$

We can generalize this to:

– Different representations of $SU(2)$:

$$\Phi(x) = \begin{pmatrix}\varphi_1(x) \\ \vdots \\ \varphi_n(x)\end{pmatrix} \quad n = 2J+1$$

(Representation of $SU(2)$ with "spin" J)

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - V(\Phi^\dagger\Phi)$$
$$= \sum_{i=1}^{n}(\partial_\mu\varphi_i^*)(\partial^\mu\varphi^i) - V\left(\sum_i \varphi_i^*\varphi\right)$$

[this "spin" is an "internal" quantum number, has nothing to do with space-time rotations]

The transformation symmetry is realized by an $n\times n$ matrix:

$$\Phi'(x) = U\Phi(x) \Leftrightarrow \begin{pmatrix}\varphi_1(x) \\ \vdots \\ \varphi_n(x)\end{pmatrix}' = \begin{pmatrix} & & \\ & n\times n & \\ & & \end{pmatrix}\begin{pmatrix}\varphi_1(x) \\ \vdots \\ \varphi_n(x)\end{pmatrix}$$

U is now an $n \times n$ matrix (so not an SU(2)) matrix but the algebra of infinitesimal transformations is the same:

$$U = \exp\left(\sum_{a=1}^{3} i\theta_a T_a \right)$$

$\{T_a\}$: 3 $n \times n$ matrices satisfying the same algebra as the τ_a:

$$\boxed{[T_a, T_b] = i\,\epsilon_{abc} T_c}$$

this makes it a representation of SU(2)

$$\delta \Phi = i \sum_a \theta_a T_a \iff \delta \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} = i \sum_{a=1}^{3} \theta_a \Big(T_a \Big)_n \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$$

Notice the # of generators is still 3.

3: dimension of the group ≡ # of infinitesimal generators ≡ dimension of the associated Lie algebra. For SU(2) it is 3

n: dimension of the representation in which the field Φ transforms ≡ size of the matrices which represent the generators

For SU(2): $n = 2J+1$ $\quad J$ = half-integer

- We can do this for a generic group G:

G = set of elements endowed with an operation $g_1 \cdot g_2$ such that:
$$\begin{cases} \exists\, \mathbb{1} : & g \cdot \mathbb{1} = \mathbb{1} \cdot g = g \\ \exists\, g^{-1} : & g^{-1} g = g g^{-1} = \mathbb{1} \end{cases}$$

- G is abelian if $g_1 g_2 = g_2 g_1 \ \forall g_1, g_2$ (e.g. $U(1)$)
- Otherwise G is non-abelian (e.g. $SU(2)$)

For example we can consider:

$$\Phi = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_N(x) \end{pmatrix} \longrightarrow U \Phi$$

U = Unitary $N \times N$ matrix: they form the group $SU(N)$ ($U^\dagger U = \mathbb{1}$, $\det U = 1$)

- Just like for $SU(2)$, the infinitesimal transformations define a Lie Algebra A_G

$$U \simeq \mathbb{1} + \omega \qquad \omega = \sum_{a=1}^{D} \lambda_a \tau_a \in A_G$$

$\tau_a \hookrightarrow$ $N \times N$ matrices

D = dimension of the algebra.

For $SU(N)$: $\boxed{D = N^2 - 1}$ (show it as an exercise)

$$[\tau_a, \tau_b] = \sum_{c=1}^{D} i f_{abc} \tau_c$$

$f_{abc} \hookrightarrow$ "structure constants" e.g. for $N=2$; $D=3$, $f_{abc} = \epsilon_{abc}$

- the numbers f_{abc} completely characterize the algebra
- We can have <u>different</u> <u>representations</u> of the same algebra: take D <u>$n \times n$</u> matrices T_a satisfying:

$$[T_a, T_b] = i f_{abc} T_c$$

$\rightarrow$ <u>same</u> f_{abc} as for the τ_a's

$\Rightarrow \exp\left(\sum_{a=1}^{D} i\lambda_a T_a\right)$ are $n \times n$ matrices acting on an n-tuplet field $\phi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$ $(n \neq N)$

- $n = N$: <u>fundamental</u> representation
- $n = D$: <u>adjoint</u> representation

For $SU(2)$:

- the fundamental representation is the $J = 1/2$:

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \tau_a = \sigma_a/2 = \left\{ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

- the adjoint representation has $n=3 \Rightarrow J=1$

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \ \underline{\text{Real}}, \quad T_a = \left\{ i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Local non-abelian invariance

take $\mathcal{L}$ invariant under some global non-abelian group G : e.g. for $SU(2)$

$$\Phi = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \qquad \Phi \longrightarrow U\Phi$$

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi^\dagger \phi)$$

Can we make $\mathcal{L}$ invariant under local group transformations?

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \longrightarrow \Big(U(x) \Big) \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

$$V(\phi^\dagger \phi) \longrightarrow V\big(\phi^\dagger \underbrace{U^\dagger U}_{\mathbb{1}} \phi\big) = V(\phi^\dagger \phi)$$

invariant.

$$(\partial_\mu \phi^\dagger)(\partial^\mu \phi) \longrightarrow \partial_\mu (\phi^\dagger U^\dagger) \partial^\mu (U\phi) =$$

$$= \left[(\partial_\mu \phi^\dagger) U^\dagger + \phi^\dagger \partial_\mu U^\dagger \right] \left[U \partial^\mu \phi + (\partial^\mu U)\phi \right]$$

$$= (\partial_\mu \phi^\dagger) \cancelto{\mathbb{1}}{U^\dagger U} (\partial_\mu \phi)$$

$$+ \phi^\dagger (\partial_\mu U) U \partial^\mu \phi + (\partial_\mu \phi)^\dagger U (\partial^\mu U) \phi + \phi^\dagger (\partial_\mu U)^\dagger (\partial^\mu U) \phi$$

kinetic term not invariant.

Add a gauge Field A_μ such that :

$$\Phi \to U(x)\Phi$$

$$A_\mu \to U(x) A_\mu U^{-1}_{(x)} - \frac{i}{g}\overbrace{(\partial_\mu U_{(x)}) U^{-1}_{(x)}}^{\text{Matrix}}$$

$g \equiv$ gauge coupling constant

replace $\partial_\mu \Phi$ by $\boxed{D_\mu \Phi = \partial_\mu \Phi - i g A_\mu \Phi}$

- A_μ has to be a matrix

$$\underbrace{(\partial_\mu U_{(x)}) U^{-1}_{(x)}}_{\displaystyle ||} \qquad \text{write } U_{(x)} = \exp(i\omega(x))$$

$$i\sum_a (\partial_\mu \alpha^a_{(x)}) T^a \qquad \omega(x) \in \text{Lie Algebra of } G$$

$$\omega(x) = \sum_{a=1}^{\dim G} \alpha^a_{(x)} T^a$$

$\alpha^a_{(x)}$: Real function

T^a: basis for the Lie Algebra

$$SU(2) : T^a = \tau^a = \frac{\sigma^a}{2}$$

$$a = 1, 2, 3$$

$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g}\sum_a i(\partial_\mu \alpha^a_{(x)}) T^a$$

= something + linear combination of Lie algebra elements

$\Rightarrow A_\mu$ must be a matrix in the Lie Algebra of G

$$\boxed{A_\mu(x) = \sum_{a=1}^{21} \underbrace{A_\mu^a(x)}_{\text{vector fields}} T^a}$$

$SU(2): \; T^a = \sigma^a/2$

$A_\mu^1(x), \; A_\mu^2(x), \; A_\mu^3(x)$ 3 vector fields

$$A_\mu(x) = \frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA^2 \\ A_\mu^1 + iA_\mu^2 & -A^3 \end{pmatrix}$$

<u>Covariant derivative</u>:

$$D_\mu \phi = \begin{pmatrix} \partial_\mu \varphi_1 \\ \partial_\mu \varphi_2 \end{pmatrix} - ig\frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA^2 \\ A_\mu^1 + iA_\mu^2 & -A^3 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$= \left(\mathbb{1}\partial_\mu - \frac{ig}{2}A_\mu\right)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

- Infinitesimal gauge transformation

$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g}\sum_a{}' \left(\partial_\mu \alpha^a(x)\right) T^a$$

$$U = \exp\left[i\sum_{a=1}^{D} \alpha^a(x) T^a\right] \underset{\alpha^a(x) \ll 1}{\simeq} \mathbb{1} + i\sum_a{}' \alpha^a(x) T^a$$

$$U A_\mu U^{-1} = \left(\mathbb{1} + i\sum_a{}' \alpha^a T^a\right)\left(\sum_b{}' A^b_\mu T^b\right)\left(\mathbb{1} - i\sum_c{}' \alpha^c T^c\right)$$

$$= \sum_b{}' A^b_\mu T^b + i\sum_{b,a}{}' \alpha^a A^b_\mu T^a T^b - i\sum_c \alpha^c A^b_\mu T^b T^c + O(\alpha^2)$$

$$= \sum_b{}' A^b_\mu T^b + i\sum_{a,b}{}' \alpha^a A^b_\mu \left[T^a, T^b\right]$$

$$\left[T^a, T^b\right] = i\sum_c{}' f^{abc} T^c \qquad f^{abc} = \text{structure constants of the Lie Algebra}$$

$$= \sum_b{}' A^b_\mu T^b - \sum_{a,b,c}{}' \alpha^a A^b_\mu f^{abc} T^c$$

$$A'_\mu = \sum_b{}' A^b_\mu T^b - \sum_{a,b,c}{}' \alpha^a A^b_\mu f^{abc} T^c + \frac{1}{g}\sum_b{}' \left(\partial_\mu \alpha^b(x)\right) T^b$$

$$\parallel$$

$$\sum_b{}' A^{b\prime}_\mu T^b$$

$$\boxed{A_\mu^{b\,\prime} = A_\mu^b + \underbrace{\sum_{a,c=1}^{D} \alpha^a A_\mu^c f^{acb}}_{\text{homogeneous}} + \frac{1}{g}\,\underbrace{\partial_\mu \alpha^b(x)}_{\text{non-homogeneous}}}$$

(Recall: $U(1)$: $\quad A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha$)

$SU(2)$: $\quad f^{abc} = \epsilon^{abc} \qquad \dim G = 3$

$$[\tau^a, \tau^b] = i \sum_c \epsilon^{abc} \tau^c \qquad \tau^a = \frac{\sigma^a}{2}$$

Local invariance under a non-Abelian group G

$\Rightarrow$ Add $\quad D = \dim G$ vector fields

$A_\mu^a \qquad a = 1 \dots D$

Kinetic term for A_μ

$U(1)$: $\quad \mathcal{L}_{A_\mu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

try:
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \qquad a = 1 \ldots \dim G$$

$$\sum_a F^a_{\mu\nu} F^{\mu\nu\, a}$$

? is it locally G-invariant?

NO! (try it: $A' = UAU^{-1} - \frac{i}{g}(\partial U)U^{-1}$)

Correct thing to do (Yang-Mills '54)

$$\underbrace{F_{\mu\nu}}_{\text{matrix}} = \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{\text{linear in } A_\mu} - \underbrace{ig[A_\mu, A_\nu]}_{\text{Quadratic in } A_\mu}$$

$[A_\mu, A_\nu]$: 0 for abelian group

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{2} \underbrace{\mathrm{Tr}}_{\text{Lie Algebra}} F_{\mu\nu} F^{\mu\nu}}$$

$$F_{\mu\nu} = \sum_a F^a_{\mu\nu}(x) T^a$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \sum_{b,c=1}^{D} f^{abc} A^b_\mu A^c_\nu$$

$$\left[SU(2) \quad = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \sum_{b,c=1}^{3} \epsilon^{abc} A^b_\mu A^c_\nu \right]$$

$$-\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{\mu\nu a} =$$

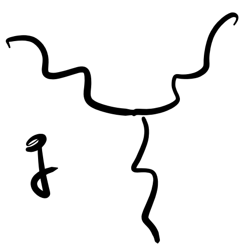
$$\mathcal{L}_{YM} = -\frac{1}{4}\sum_a{}' \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 \longleftarrow \text{Free}$$

$$g\sum_{a,bc}{}' \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\epsilon^{abc}A^{b\mu}A^{c\nu} +$$

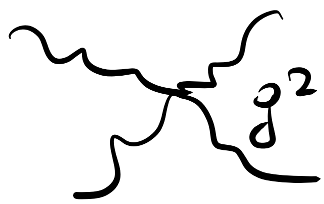
$$\frac{1}{4}g^2\sum_{\substack{abc\\de}}{}' f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}$$

interactions

$(g = 0 \Rightarrow$ no interactions$)$



g



g^2

<u>Rules to write a non-Abelian gauge theory:</u>

- Choose the group(s)
- Choose a coupling constant for each group
- choose the representations for matter fields
- choose <u>gauge-invariant</u> interactions between matter fields

Higgs Mechanism

It works like in the abelian case, except that the symmetry breaking can be partial:

- if $U \Phi_0 \neq \Phi_0 \quad \forall U \Rightarrow$ Symmetry is completely broken

ground state

$\Rightarrow$ All gauge fields $A_\mu^a \quad a = 1 \dots D$ become massive

- if $U \Phi_0 = \Phi_0$ for some U's

$\Rightarrow$ some gauge fields stay massless.

More precisely: the $U \in G$ which leave Φ_0 invariant form a subgroup H.

we can split the generators of G in two classes:

$$\{T_a\}_{a=1}^{D} = \{\underbrace{T_1^{(H)} \dots T_d^{(H)}}_{\text{generators of } H} ; \underbrace{T_{d+1}^{G/H} \dots T_D^{G/H}}_{\text{the rest of the generators}}\}$$

- $T_a^{H} \Phi_0 = 0 \Rightarrow$ the corresponding gauge fields are massless

$\Rightarrow \begin{cases} d \text{ massless} \\ D-d \text{ massive} \end{cases} A_\mu^a$ e.g. $SO(3) \longrightarrow SO(2)$ with $\Phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$