Electro-weak theory

> Electromagnetism + weak interactions

- Gauge Group:

$$\underset{\swarrow\ \text{3 vectors}}{SU(2)} \; \underset{\searrow\ \text{the two groups commute with each other}}{\times} \; \overset{\nearrow\ \text{1 Abelian vector}}{U(1)}$$

- Coupling constant

$$SU(2) \longrightarrow g$$

$$U(1) \longrightarrow g'$$

- Matter $\left\{ \begin{array}{l} \text{1) a complex } SU(2)\text{-doublet } H \text{ of scalar fields (fundamental rep. of } SU(2)) \\ \text{2) Fermions} \end{array} \right.$

$$\text{Fermions} \swarrow \text{Quarks} \qquad \searrow \text{Leptons}$$

Bosonic Sector of EW theory

(W^a_μ, B_μ, H)

W^a_μ : SU(2)

B_μ : U(1)

H : Higgs doublet

$$SU(2)\begin{cases} H \longrightarrow UH \\ W_\mu \longrightarrow U W_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \\ B_\mu \longrightarrow B_\mu \end{cases}$$

$$U(1)\begin{cases} H \longrightarrow e^{i\frac{\alpha(x)}{2}} H \\ W_\mu \longrightarrow W_\mu \\ B_\mu \longrightarrow B_\mu + \frac{1}{g'}\partial_\mu \alpha(x) \end{cases}$$

(U(1) charge of H is 1/2)

Covariant derivative :

$$D_\mu H = \Big(\partial_\mu H - ig\underbrace{\sum_a W^a_\mu \tau^a}_{W^1\tau^1 + W^2\tau^2 + W^3\tau^3} H - \frac{ig'}{2} B_\mu \mathbb{1} H\Big)$$

charge of H = 1/2

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$D_\mu \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \partial_\mu h_1 \\ \partial_\mu h_2 \end{pmatrix} - i\frac{g}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W^3 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$-i\frac{g'}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \partial_\mu h_1 \\ \partial_\mu h_2 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\mathcal{L}^{EW}_{\text{Bosons}} = \underset{\text{YM for } SU(2)}{-\frac{1}{4}\sum_a W^a_{\mu\nu}W^{\mu\nu a}} \underset{\text{Maxwell for } U(1)}{-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}}$$

$$+(D_\mu H)^\dagger(D^\mu H) \underset{\substack{\uparrow \\ \text{Wrong sign!}}}{+} m^2 \underbrace{H^\dagger H - \lambda(H^\dagger H)^2}_{\substack{SU(2) \text{ and } U(1) \\ \text{invariant}}}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\sum_{b,c}\epsilon^{abc}W^b_\mu W^c_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

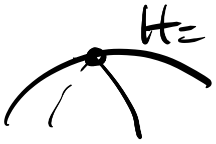
Higgs potential

$$V(H) = -m^2 H^\dagger H + \lambda (H^\dagger H)^2$$

- Classical vacua : (extrema of V)

  - $H = 0$, maximum of V

    ($V \underset{H\simeq 0}{\simeq} -m^2 H^\dagger H$



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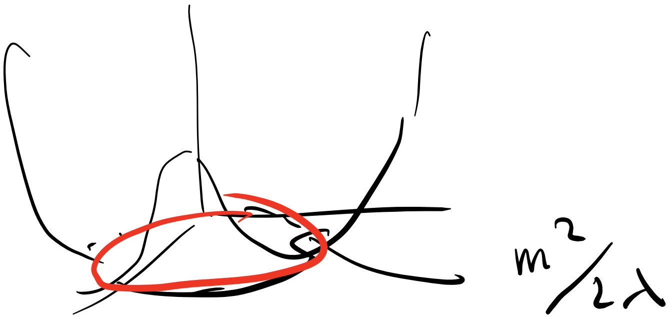
  - $H^\dagger H = \dfrac{m^2}{2\lambda}$ family of minima



$$\left(\frac{\partial V}{\partial H} = \frac{\partial V}{\partial (H^\dagger H)} \frac{\partial H^\dagger H}{\partial H} = H^\dagger \frac{\partial V}{\partial (H^\dagger H)} = 0 \right.$$

$$\frac{\partial V}{\partial H^\dagger} = \cdots \frac{\partial V}{\partial H^\dagger H} H^\dagger = 0$$

$$\Leftarrow 0 = \frac{\partial V}{\partial (H^\dagger H)} = -m^2 + 2\lambda H^\dagger H$$

$$\Rightarrow \boxed{H^\dagger H = \frac{m^2}{2\lambda}} \quad \text{extremises } V(H) \;)$$

minima

$$v = \sqrt{\frac{m^2}{\lambda}} \qquad H^\dagger H = \frac{v^2}{2}$$

stable vacuum

$\Rightarrow$ $SU(2) \times U(1)$ symmetry is broken by the vacuum expectation value of H

$$\text{Choose} \quad H = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \leftarrow H^\dagger H = v^2/2$$

$$(\mathrm{Re}\, h_1 = \mathrm{Im}\, h_1 = \mathrm{Im}\, h_2 = 0)$$

$$U(1): \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \longrightarrow e^{i\frac{\alpha(x)}{2}} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\frac{i\alpha}{2}} \frac{v}{\sqrt{2}} \end{pmatrix} \neq \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

(not invariant under $U(1)$)

$$SO(2): \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \rightarrow \underset{\underset{\text{Arbitrary}}{\uparrow}}{U} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$\forall\ U \in SU(2) \neq \mathbb{1}$$

Infinitesimally:

$$U^{(2)} = \mathbb{1} + i \sum_{a=1}^{3} \alpha^a(x) \tau^a \qquad SU(2)$$

$$U^{(1)} = \mathbb{1} + i \frac{\alpha(x)}{2} \qquad U(1)$$

$$H' = \underset{\leftrightarrow}{U^{(1)} U^{(2)}} H = \left(1 + i \sum \alpha^a \tau^a\right)\left(1 + \frac{i\alpha}{2}\right) H$$

to linear order in α's

$$\Rightarrow H' = H + \Big(\underbrace{i \sum_a \alpha^a(x) \tau^a + \frac{i}{2} \alpha(x) \mathbb{1}}_{2\times 2 \text{ matrix}}\Big) H$$

$$H' = H \quad (\text{invariance})$$

$$\Leftrightarrow \delta H = \left(i \sum_a \alpha^a(x) \tau^a + \frac{i}{2} \alpha(x) \mathbb{1}\right) H = 0$$

Pure SU(2) transformation

$$\Rightarrow \alpha = 0$$

$$\delta H = i \sum_a \alpha^a(x) \tau^a H$$

$$H_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \qquad \frac{\sigma}{2}^1 H_0 = \frac{1}{2}\begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{\sigma}{2}^2 H_0 = \frac{1}{2}\begin{pmatrix} i\,v/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{\sigma}{2}^3 H_0 = \frac{1}{2}\begin{pmatrix} 0 \\ -v/\sqrt{2} \end{pmatrix}$$

$$\delta H = \frac{i}{2}\begin{pmatrix} (\alpha^1 + i\alpha^2)\,v/\sqrt{2} \\ -\alpha^3\, v/\sqrt{2} \end{pmatrix}$$

$$= 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

only $\mathbb{1} \in SU(2)$ leaves H_0 invariant

- Pure U(1) $\quad \alpha^a = 0 \;,\; \alpha \neq 0$

$$\delta H = \frac{i}{2}\alpha\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0 \Rightarrow \alpha = 0$$

• Combined SU(2) × U(1) transformation

$$\delta H = \left(i \sum_a \alpha^a(x) \tau^a + \frac{i}{2} \alpha(x) \mathbb{1} \right) \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$= \frac{i}{2} \begin{pmatrix} (\alpha^1 + i\alpha^2) v/\sqrt{2} \\ -\alpha^3 \frac{v}{\sqrt{2}} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 \\ \alpha \frac{v}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{i}{2} \begin{pmatrix} (\alpha^1 + i\alpha^2) v/\sqrt{2} \\ (\alpha - \alpha^3) v/\sqrt{2} \end{pmatrix} = 0$$

$\alpha^1 = \alpha^2 = 0$, $\boxed{\alpha = \alpha^3} \neq 0$!

A subgroup of SU(2) × U(1) is unbroken

it is generated by $\tau^3 + \mathbb{1}_{U(1)}$

$$U = \underbrace{e^{i\alpha\tau^3}}_{SU(2)} \underbrace{e^{i\frac{\alpha}{2}}}_{U(1)} \qquad U H_0 = H_0$$

these transformations form a $\underline{U(1)}$ $\underline{\text{subgroup}}$ of $SU(2) \times U(1)$

$$SU(2) \times U(1) \xrightarrow[\text{spontaneously broken}]{} U(1)_{EM}$$

$$\text{Expect: } \begin{cases} 3 \text{ massive vectors} & W^{\pm}_{\mu}, Z_{\mu} \\ 1 \text{ massless vector} & A_{\mu} \\ 1 \text{ massive real scalar, neutral under } U(1)_{EM} & h \end{cases}$$

Gauge boson masses (Abridged)

$$\mathcal{L}_{mass} = (D_\mu H)^\dagger (D^\mu H)\Big|_{H=\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}} =$$

$$= \left[-\frac{i}{2}\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}\right]^\dagger \times$$

$$\left[-\frac{i}{2}\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}\right] =$$

$$= g^2\frac{v^2}{8}\left[W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu} + \left(g'/g\, B_\mu - W^3_\mu\right)^2 \right]$$

$$\mathcal{L}_{\substack{\text{Free}\\ \text{Vectors}}} = -\frac{1}{4}\left(W^1_{\mu\nu}W^{1\,\mu\nu} + W^2_{\mu\nu}W^{2\,\mu\nu} + W^3_{\mu\nu}W^{3\,\mu\nu}\right)$$

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \mathcal{L}_{\text{mass}}$$

θ_W: Weinberg Angle

Define: $\tan\theta_W \overset{\text{def}}{\equiv} g'/g$

$$\begin{cases} Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu = \sin\theta_W W^2_\mu + \cos\theta_W B_\mu \end{cases}$$

$$\begin{cases} B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu \\ W^3_\mu = \sin\theta_W A_\mu + \cos\theta_W Z_\mu \end{cases}$$

Substitute in $\mathcal{L}_{\text{Fee}}$ $\quad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu ,$

$W^\pm_\mu = W^1_\mu \mp i W^2_\mu \quad W^\pm_{\mu\nu} = \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L}_{kin} = -\frac{1}{4}\left(Z_{\mu\nu}Z^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}\right) +$$

$$-\frac{1}{2}W^+_{\mu\nu}W^{\mu\nu\,-} + \mathcal{L}_{\text{mass}}$$

$$\mathcal{L}_{\text{mass}} = m_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}m_Z^2 Z^\mu Z_\mu$$

$$m_W = g\frac{V}{2} \qquad m_Z = \frac{m_W}{\cos\theta_W}$$

Electric Charge

$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha$$

under an unbroken gauge transformation

$$A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu$$

$U(1)_{EM}$ = SU(2) in the 3rd direction + U(1) transformation with same parameter

$$\begin{cases} \delta W^3_\mu = \frac{1}{g}\partial_\mu \underset{\parallel}{\alpha^3} + \underbrace{\sum_{b,c} \epsilon^{3bc} \overset{1,2}{\underset{0}{\underset{\parallel}{\alpha^b W^c}}}}_{\substack{\parallel \\ 0}} \\ \delta B_\mu = \frac{1}{g'}\partial_\mu \alpha \end{cases}$$

$\alpha = \alpha^3$

$$\delta A_\mu = \sin\theta_W \delta W^3_\mu + \cos\theta_W \delta B_\mu =$$

$$= \frac{\sin\theta_W}{g}\partial_\mu \alpha + \frac{\cos\theta_W}{g'}\partial_\mu\alpha =$$

$$= \frac{1}{g}\left(\sin\theta_W + \frac{g}{g'}\cos\theta_W\right)\partial_\mu \alpha =$$

$$\tan\theta_W \overset{def}{\equiv} g'/g$$

$$= \frac{1}{g}\left(\sin\theta_W + \frac{\cos\theta_W}{\sin\theta_W}\cos\theta_W\right)\partial_\mu \alpha$$

$$= \frac{1}{g\sin\theta_W}\left(\underset{\overset{\|}{1}}{\sin^2\theta_W + \cos^2\theta_W}\right)\partial_\mu \alpha$$

$$\equiv \frac{1}{e}\partial_\mu \alpha$$

$$\boxed{e = g\sin\theta_W}$$

$$\sin^2\theta_W = \frac{(g'/g)^2}{1+(g'/g)^2}$$

- $$\delta W^1_\mu = \cancel{\frac{1}{g}\partial_\mu \underset{\overset{\|}{0}}{\alpha^1}} - \sum_{b,c} \underset{\overset{\|}{+\,\alpha^3 W^2}}{\epsilon^{1\underset{3}{b}\underset{2}{c}}\alpha^b W^c}$$

$\alpha^3 = \alpha\,,\ \alpha^1 = \alpha^2 = 0$

- $$\delta W^2_\mu = \cancel{\frac{1}{g}\partial_\mu \alpha^2} - \alpha^3 W^1$$

$$\Rightarrow \begin{cases} \delta W^+_\mu = i\alpha^3 W^+_\mu \\ \delta W^-_\mu = -i\alpha^3 W^-_\mu \end{cases} \qquad \alpha^3 = \alpha$$

$$W^\pm_\mu \longrightarrow e^{\pm i\alpha \underset{Q}{\overset{\|}{1}}} W^\pm_\mu$$

$W_\mu^\pm$ have charges $\pm e$.

- $\delta Z_\mu = 0$ (exercise)

  Z_μ is a neutral vector

- <u>Higgs boson Lagrangian</u>

$$H = \begin{pmatrix} 0 \\ v/\sqrt{2} + \frac{h(x)}{\sqrt{2}} \end{pmatrix}$$

$h(x)$: physical Higgs boson

$h(x)$ is <u>neutral</u> under $U(1)_{Em}$: Under $U(1)_{Em}$:

$$\begin{pmatrix} 0 \\ H \end{pmatrix} \to \begin{pmatrix} 0 \\ H \end{pmatrix}$$

$$\Rightarrow \delta h = 0$$

(exercise:)

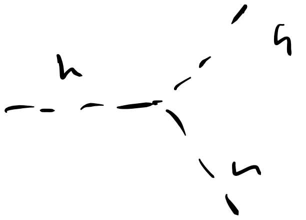
$$\underline{\mathcal{L}} = \frac{1}{2}\partial_\mu h \, \partial^\mu h - \frac{1}{2}\underline{m_h^2} h^2 + g\frac{m_h^2}{4 m_W} h^3$$

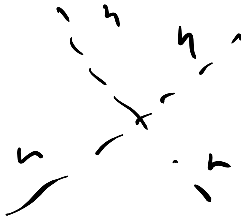
$$- \boxed{\frac{g^2}{32}\frac{m_h^2}{m_W^2}} h^4 \longrightarrow \text{not observed yet}$$

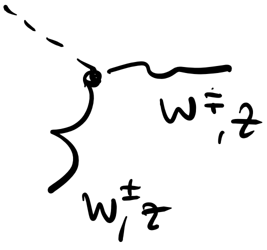
$$\boxed{m_h = \sqrt{2\lambda}\, v}$$

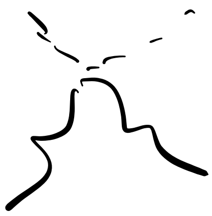
$$+ \frac{2h}{v}\left(m_W^2 W_\mu^+ W^{\mu +} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \right)$$

$$+ h^2/v \left(m_W^2 W_\mu^+ W^{\mu +} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \right)$$









Fermions

$$G = SU(2)_L \times U(1)_Y$$

- Electron-type $\longrightarrow$ $\Psi = \psi_L + \psi_R$
- Neutrino-type $\longrightarrow$ ν_L only

$$\psi_L = \frac{1-\gamma^5}{2}\psi \;, \quad \psi_R = \frac{1+\gamma^5}{2}\psi$$

$$\gamma^5 \psi_L = -\psi_L \qquad \gamma^5 \psi_R = \psi_R$$

only left-handed fermions transform under $SU(2)_L$!

$$\psi_L \equiv e_L \;, \; \nu_L \rightarrow$$

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

- doublet under $SU(2)_L$
- $Y_L = -\frac{1}{2}$ under $U(1)_Y$

e_R:
- singlet under $SU(2)$ (invariant)
- $Y_R = -1$

under an $SU(2)_L \times U(1)_Y$ transformation:

$$L \to e^{-i\frac{\alpha}{2}} U L$$

$$e_R \to e^{-i\alpha} e_R$$

· infinitesimally:

$$\delta L = \left(-i\frac{\alpha}{2}\mathbb{1} + i\sum_{a=1}^{3}\alpha^a \tau^a\right) L$$

$$\delta e_R = -i\alpha e_R$$

· Under $U(1)_{EM}$ $(\alpha = \alpha^3,\ \alpha^1 = \alpha^2 = 0)$

$$\delta L = \left(-i\frac{\alpha}{2}\mathbb{1} + i\alpha\frac{\sigma^3}{2}\right) L \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$Y + I_3$

$$= +\frac{i}{2}\begin{pmatrix}-\alpha+\alpha & 0\\ 0 & -\alpha-\alpha\end{pmatrix} L = -i\alpha\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} L$$

$$L = \begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \Rightarrow \begin{cases}\delta\nu_L = 0\\ \delta e_L = -i\alpha e_L\end{cases}$$

$$\left.\begin{array}{l}\nu_L \text{ is uncharged}\\ e_L \text{ has charge } -1\end{array}\right\} \text{ under } U(1)_{EM}$$

· $\delta e_R = -i\alpha e_R$ under $U(1)_{EM}$

$\Rightarrow e_R$ has charge -1

$e_L + e_R = \psi_e$ has charge -1 under EM.

(can write down QED)

$$\boxed{Q = \underset{\underset{SU(2)}{\downarrow}}{I^3} + \underset{\underset{U(1)}{\searrow}}{Y}}$$

I^3 is the eigenvalue of τ^3

For e: $I^3 = -1/2$, $Y = -1/2 \Rightarrow Q_e = -1$

ν : $I^3 = +1/2$, $Y = -1/2 \Rightarrow Q_\nu = 0$

$\vdots$

$$\mathcal{L}_{kin} = i\bar{L}\underset{\underset{SU(2)_L \times U(1)_Y}{\uparrow}}{\not{D}}L + i\bar{e}_R\underset{\underset{\text{only } U(1)_Y}{\nearrow}}{\not{D}}e_R$$

$$\not{D}L = \gamma^\mu\left(\partial^\mu - ig\sum_a W^a_\mu \tau^a - ig'\overset{-1/2}{Y_L}B_\mu\right)L$$

$$\not{D}e_R = \gamma^\mu(\partial^\mu - ig'\overset{-1}{Y_e}B_\mu)e_R$$

Fermion masses

Just add

$$\cancel{m_e \bar{\psi}_e \psi_e} = m_e(\bar{e}_L e_R + \bar{e}_R e_L)$$

WAIT!

$\bar{e}_L$: part of SU(2) Doublet

e_R: SU(2) singlet!

not $SU(2)_L$ invariant!

(neither $U(1)_Y$ invariant)

⇒ an explicit mass term breaks $SU(2)_L \times U(1)_Y$ explicitly

Add instead.

$$\mathcal{L}_{hLe_R} = -y_e \bar{L} H e_R + h.c.$$

under SU(2)

$$\bar{L}\cancel{U^+} \quad \cancel{U}H \quad e_R$$

$SU(2)_L$ invariant

$\times U(1)_Y$ invariant

$\bar{L}$ has charge $+1/2$

H " " $+1/2$

e_R " " -1

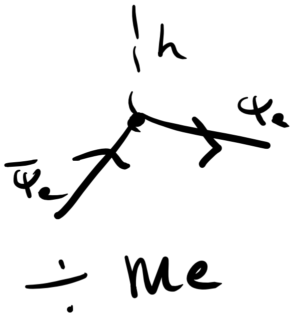
$$\mathcal{L} = -y_e \left(\bar{\nu} \ \ \bar{e}_L\right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}} \end{pmatrix} e_R =$$

$$= -y_e \frac{v}{\sqrt{2}} \bar{e}_L e_R - y_e \frac{vh}{2} \bar{e}_L e_R$$

$$+ \ h.c.$$

$$= -\frac{y_e \enclose{circle}{v}}{\sqrt{2}} \left(\bar{\psi}_e \psi_e\right) - \frac{y_e v}{2} h \bar{\psi}_e \psi_e$$

$$\boxed{m_e = -\frac{y_e v}{\sqrt{2}}}$$



$$\left(m_\nu = 0\right)$$

All mass terms in the SM come from $v \neq 0$

$$\times 3 \quad \begin{array}{ccc} y_e & y_\mu & y_\tau \\ L_e, & L_\mu, & L_\tau \\ e_R, & \mu_R, & \tau_R \end{array} \qquad m_i = \frac{v y_i}{\sqrt{2}}$$