Contents

Chapter 3

Scattering and the S-matrix

3.1 Introduction

Scattering processes consist of one of the main components of these lecture notes, with the aim of providing all the ingredients necessary for the theoretical calculation of quantities that can be compared with experimental data. What we call a *scattering process* corresponds either to the collision of a beam of particles on a fixed target, or to the collision of two beams of particles. In both cases, the effect of the emerging final state can be observed in a detector, and the observations can then be compared with theoretical predictions. Whereas technically we could imagine other processes involving more than two particles in the initial state, these processes are irrelevant as they cannot be set up experimentally from a practical point of view. On the other hand, decay processes in which one particle turns into several other particles can also be seen, in QFT, as scattering processes. Therefore, we focus from now on on processes involving either one or two initial-state particles, and an arbitrary (but finite) number of final-state particles.

Scattering has played a major role in the development of subatomic physics. For instance, it yielded the discovery of the structure of the atom, that of the nucleus, and that of the elementary particles. Moreover, it allowed for the study of the properties of these objects, hence sharpening our understanding of the microscopic world. Nowadays scattering experiments using very high energy accelerators, like the currently operating Large Hadron Collider at CERN, are used to probe fundamental interactions and to shed more light on how our universe functions at its most fundamental level. In practice, this relies on the fact that quantum field theory provides extremely powerful methods to handle scattering computations, and it hence allows physicists to achieve theoretical predictions as precise as currently required by experimental data. Consequently, theoretical models can be compared with data, and conclusive statements about the phenomenological viability of different modelings of the dynamics of the microscopic world can be drawn.

Scattering-related calculations in QFT involve a primary quantity known as the *S-matrix*, whose elements can be seen as projections of an initial state $|i\rangle$ onto a final state $|f\rangle$, or equivalently as the transition amplitude from the state $|i\rangle$ to the state $|f\rangle$. The calculation of these matrix elements is based on the concept of *asymptotic states*, that correspond to states produced by creation operators $a^\dagger_{\mathbf{p}}$ at asymptotically early or late times relative to the time at which scattering occurs. Modern computational techniques additionally rely on the well-known method of *Feynman diagrams* and *Feynman rules* [19], two concepts that bear the name of Richard Feynman (1918 – 1988). These techniques can be derived from analytical expressions for the elements of the S-matrix, and they provide a simple and elegant way to handle sometimes complicated QFT scattering calculations. Moreover, Feynman diagrammatic methods are strongly connected to perturbative expansions in QFT, as they allow for the calculation of quantities relevant for particle scattering from simple and systematic representative diagrammatic series expansions.

In section 3.2 we introduce a notion that is central in calculations associated with scattering processes, and that can easily be measured experimentally: the *cross section*. Though this quantity has the dimensions of an area, it has a more abstract meaning and can be thought of as a measure of the strength or the likelihood of a specific scattering process. Here, we consider as a specific process a collision in which both the initial state and the final state are well defined. In other words, a cross section is a measure

of the probability that a specific process takes place. Large cross sections are associated with frequently occuring processes, whereas small cross sections are related to rarer processes. We first characterise cross sections in the non-relativistic case, which allows us to introduce useful physical concepts such as Green's function and the S-matrix. We next develop this in the relativistic case using quantum field theory methods, and we derive a Lorentz-invariant expression for cross sections in QFT. A specific emphasis is put on $2 \to 2$ processes in which two particles annihilate into two possibly different particles. As an illustration, we make use of the case of electromagnetic interactions to demonstrate through a simple example how QFT predictions reproduce non-relativistic quantum mechanical predictions in the non-relativistic limit. This example additionally allows us to briefly introduce vector fields, and the concept of *propagators* that we rigorously treat later in the following sections (the full treatment being unnecessary for the illustrative cross section calculation achieved in section 3.2).

Cross sections can also be connected to the rate at which a decay process occurs. This is when we consider decay processes as $1 \to n$ scattering processes in which one initial-state particle decays into a set of n other particles. This situation is detailed in section 3.3, in which we provide formulas for *differential decay rates* and partial and total *decay widths* of a particle. The formulas are next applied to the case of the simplest class of decay processes, that in which one heavy particle decays into two lighter particles. In this situation, the partial decay width takes a compact and generic form that can be easily applied to any $1 \to 2$ decay.

In order to design a method allowing for calculations of cross sections and decay rates in a straightforward manner (which is the goal of this chapter), we go back to the general definition of an element of the S-matrix in section 3.4. We next derive the so-called *Lehmann-Symanzik-Zimmermann (LSZ) reduction formula* [20], that forms the cornerstone of any scattering calculation in QFT. The strength of the LSZ formula is that it relates matrix elements to quantum fields. It, moreover, consists of a gateway to Feynman rules and Feynman diagrams, whose connection to scattering calculations originates from the LSZ formula. This is further handled in detail in sections 3.5 and 3.6.

In section 3.5, we apply the LSZ formula to a specific calculation. We focus on the case of a real scalar field whose equation of motion is the Klein-Gordon equation, and that we explored in section 2.5. We make use of the LSZ formula to calculate the quantity known as the *Feynman propagator* of the Klein-Gordon field, that describes how a real scalar particle can evolve from a point x in space-time to another point y. We derive the associated Feynman rule that will be used further in these lecture notes. Through this example, we relate calculations in the free theory to calculations in the interacting theory. In the former case, there is no interaction (and thus no scattering) but fields are easier to manipulate. On the contrary, in the interacting case (that is the interesting case in light of describing the laws of nature), fields are objects more complicated to manipulate because of the impact of the interactions. The derivation of the connection between the two introduces the concept of *interaction-picture fields* and that of *interaction-picture propagators*. This last object dictates time evolution in the presence of interactions and is central in the proof showing that the calculation of any scattering amplitude can be handled through Feynman diagrams built from the interaction Lagrangian of the physics model. We close section 3.5 by generalising the results obtained for the Feynman propagator to the case of matrix elements involving an arbitrary (but finite) number of fields.

The obtained generalised result is next used in section 3.6 to elaborate on the notion of Feynman rules and Feynman diagrams, to connect them to the elements of the S-matrix (and thus to cross sections and decay rates), and to explain the origin of the diagrammatic way proposed by Feynman to calculate any scattering amplitude in QFT. This makes use of *Wick's theorem* [21], that bears the name of Gian-Carlo Wick (1909 – 1992) and that allows for the reduction of an arbitrary number of products of creation and annihilation operators to sums of products of pairs of these operators called *contractions*. This theorem naturally leads to a diagrammatic expansion for the elements of the S-matrix so that we can derive the rules that match a diagram to the associated analytical expression. These rules can be cast in two forms: The first one consists of *position-space Feynman rules*. And the second one of *momentum-space Feynman rules*. The latter is the true goal of this chapter, as it lies at the heart of almost every practical scattering calculation in QFT today. It indeed allows for a straightforward calculation of any matrix element entering the cross section and decay rate formulas derived in sections 3.2 and 3.3.

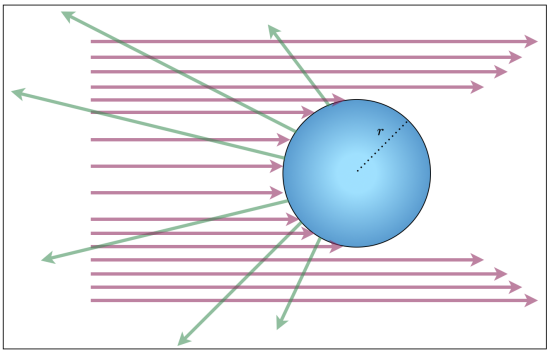


Figure 3.1: Schematic illustration of the Geiger–Marsden experiment in which a beam of alpha particles (in purple) is fired at one of the atomic nuclei (in blue) of a metallic target. In contrast with predictions from Thomson's model of the atom, some alpha particles are scattered in different directions (in green). This allowed Rutherford and his team to deduce that atoms were made of a nucleus containing most of its mass and all of its positive charge. This conclusion originates from a measurement of the cross-sectional area of the scattering object $\sigma = \pi r^2$, that is in addition proportional to the number of particles scattered.

3.2 Cross sections

3.2.1 Generalities

In order to introduce the concept of a *cross section*, we consider scattering experiments in which the probability of multiple collisions is negligible. This applies, for example, to the well known series of Geiger–Marsden experiments from the beginning of the 20$^{\text{th}}$ century, in which a beam of alpha particles was scattered from a thin metallic foil [22, 23]. These experiments used alpha particles emitted by a radioactive element as probes of the atomic world, that was unknown and unseen in those days. They were conducted by Hans Geiger (1882 – 1945) and Ernest Marsden (1889 – 1970), under the supervision of Ernest Rutherford (1871 – 1937). At this time, the prevailing model for the atomic structure was the 'plum pudding model' devised by Joseph Thomson (1856 – 1940), in which an atom is made of a positively-charged sphere containing a uniform distribution of electrons. According to this model, any beam of alpha particles fired on a target would just go through it, without being deflected. Rutherford and his team found on the contrary that beams of alpha particles were scattered in all directions, as illustrated in figure 3.1. From the measurement of the amount of alpha particles scattered, they concluded that atoms have a largely open structure with a tiny and heavy nucleus at its centre comprising all of its positive charge and most of its mass [24].

More precisely, by measuring the amount of scattered particles, Rutherford could determine the cross-sectional area σ of atomic nuclei (*i.e.* in this example gold atoms) within the metallic foil. Furthermore, if we assume that the nucleus has a radius r, then we have $\sigma = \pi r^2$. However, the cross-sectional area can also be given by the ratio of the number N of particles scattered to the total number of particles reaching the target (per unit of time). Considering that the detector covers a certain solid angle $\mathrm{d}\Omega$ with respect to the scattering centre in the direction (θ, ϕ), the *differential cross section* $\mathrm{d}\sigma/\mathrm{d}\Omega$ is defined by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{N}{F} = \frac{N}{v\,n}\,. \tag{3.2.1}$$

This expression explicitly includes a dependence on the *incoming flux*, F, defined as the number of particles crossing a unit area normal to the beam direction per unit of time. The flux F is thus given by the product of the number density n in the beam and the speed v of the particles in the beam. The *total cross section* is obtained by integrating the differential cross section over all solid angles,

$$\sigma_{\text{tot}} = \int \mathrm{d}\sigma = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos\theta\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,. \tag{3.2.2}$$

In real-life situations, the number N of scattering events is obtained by generalising the above expression in order to account for possibly time-dependent factors that depend on the details of the experiment. These include, for instance, the duration ΔT of the experiment or the number density of particles in the

target. Generically, this can be written as

$$N = \sigma_{\text{tot}} \int_{\Delta T} \mathrm{d}t\, L(t)\,, \tag{3.2.3}$$

so that the cross section stays independent of the details of the experiment. The quantity $L(t)$ is the *instantaneous luminosity*, and it depends on the properties of all objects involved in the scattering process (beams, targets, *etc.*). The integral of the luminosity with respect to time is called the *integrated luminosity* L, and it directly allows for an assessment of the amount of data collected by an experiment (*i.e.* the number of collisions that happened).

For the moment, we have considered that scattering has behaved classically, *i.e.* the particle in the beam either scatters in a given direction, or they do not scatter in that direction. In quantum mechanics there are no classical trajectories and the notion of a cross-sectional area must be generalised: it is related to the (differential) probability $\mathrm{d}\mathcal{P}/\mathrm{d}\Omega$ of scattering in a given solid angle.

The rate of occurrence that a given scattering process happens in a certain solid angle $\mathrm{d}\Omega$ is given by the associated differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ defined by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{F}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\Omega}\,. \tag{3.2.4}$$

It is proportional to the differential quantum mechanical probability of scattering $\mathrm{d}\mathcal{P}$, and to the inverse of the incoming flux F normalising the cross section such that any initial state beam or target is made of a single particle.

The purpose of the following subsections is to characterise further this cross section. We focus first on the case of non-relativistic quantum mechanics in section 3.2.2, which allows us to introduce different physical concepts on known basics. Next, we move to the relativistic case in the context of QFT (in section 3.2.3), whose non-relativistic limit should reproduce the non-relativistic quantum mechanical results. A comparison of the two approaches in the case of a specific example is made in section 3.2.4, while the rest of the chapter is dedicated to the introduction of a very powerful method to handle cross section calculations in QFT. In these notes, we will indeed discuss in detail the origins of Feynman diagrams and Feynman rules, and how these concepts can be used to simplify drastically cross section calculations.

3.2.2 Scattering of non-relativistic particles

The properties of the illustrative quantum field that we introduced in section 2.5 originate from the Klein-Gordon equation. It turns out that such a field describes the dynamics of massive spinless particles (as further elaborated in chapter **??**). For this reason, we adopt in the current and following subsections a scattering process involving only massive spinless particles.

In non-relativistic quantum mechanics, the scattering (or the collision) of two particles can be seen as a process in which the two particles are initially very far from each other, and thus free. They then approach each other to finally interact and exchange energy and momentum, before moving apart and becoming free again. In this configuration of an *elastic collision*, the nature and the number of particles is preserved. On the contrary, in *inelastic collisions* energy can be converted into mass, and conversely mass can be converted into energy. The number and the nature of the scattering particles is thus not preserved, and as already mentioned we need to rely on the framework of QFT to handle associated calculations.

In quantum mechanics, the relative motion of two scattering particles of mass m_1 and m_2 and position $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ is described by the Schrödinger equation (2.2.56) acting on a ket $|\Psi(t)\rangle$. This yields, once the Schrödinger equation is projected on the position basis $|\mathbf{x}\rangle$ and after using the correspondence principle (2.2.81),

$$i\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \left[-\frac{1}{2m}\Delta + V(\mathbf{x})\right]\Psi(\mathbf{x},t)\,. \tag{3.2.5}$$

Here we have assumed that the Hamiltonian is time-independent, and is therefore simply given by the sum of the kinetic and potential energies of the system. Moreover, we have considered that the potential

only depends on the relative position $\mathbf{x}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ of the two particles, and the parameter m is identified with the reduced mass of the system,

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}\,. \tag{3.2.6}$$

The potential being time-independent, the time-dependence of the wave function $\Psi(\mathbf{x},\, t)$ is obtained by means of the time-evolution operator (2.2.72). This gives

$$\Psi(\mathbf{x},\, t) = \psi(\mathbf{x})\, e^{-iEt}\,, \tag{3.2.7}$$

where E is the energy of the system. Consequently, the properties of the scattering process can all be determined once the *stationary wave function* $\psi(\mathbf{x})$ is known. This wave function $\psi(\mathbf{x})$ satisfies the time-independent Schrödinger equation

$$\left[-\frac{1}{2m}\Delta + V(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x})\,, \tag{3.2.8}$$

in which $\psi(\mathbf{x})$ contains both the incident and the scattered wave,

$$\psi(\mathbf{x}) = \psi_{\text{inc}}(\mathbf{x}) + \psi_{\text{scat}}(\mathbf{x})\,. \tag{3.2.9}$$

Solving (3.2.8) is therefore equivalent to expressing the (unknown) scattered wave $\psi_{\text{scat}}(\mathbf{x})$ in terms of the (known) incident wave $\psi_{\text{inc}}(\mathbf{x})$.

In practical applications the potential $V(\mathbf{x})$ is short range and satisfies

$$V(\mathbf{x}) = \mathcal{O}\left(\frac{1}{||\mathbf{x}||^3}\right) \qquad \text{for} \qquad ||\mathbf{x}|| \to \infty\,. \tag{3.2.10}$$

It can thus be neglected for $||\mathbf{x}|| \to \infty$. In other words, the incident and scattered waves correspond to *free waves in the asymptotic limit*, *i.e.* far from the scattering centre. It is worth mentioning that this asymptotic form of the wave function precisely corresponds to the quantity of interest. Observations near the scattering centre, that could be for instance an atomic nucleus, are indeed mostly not feasible. Detection instead usually takes place at large distances, where the particles should behave like free particles. The wave function $\psi(\mathbf{x})$ is therefore a solution of the free Schrödinger equation, and it should have a form similar to that of a free wave. In general, a free wave $\varphi(\mathbf{x})$ can be uniquely decomposed in terms of an *incoming wave* and an *outgoing wave*. Such a decomposition is given, in spherical coordinates, by

$$\varphi(\mathbf{x}) \equiv \varphi(r,\theta,\phi) = A_-(\theta,\phi)\,\frac{e^{-ikr}}{r} + A_+(\theta,\phi)\,\frac{e^{ikr}}{r}\,, \tag{3.2.11}$$

in which the first term in this sum represents the incoming wave and the second term the outgoing one. The coefficient functions $A_-(\theta,\phi)$ and $A_+(\theta,\phi)$ are called the *incoming scattering amplitude* and *outgoing scattering amplitude* respectively, and they only depend on the angular variables θ and ϕ. Moreover, the wave number k is related to the energy of the wave, through $E = k^2/(2m)$.

Exercise 3.1. By solving the time-independent free Schrödinger equation, demonstrate that a free wave can be generically written, using spherical coordinates, in the form

$$\varphi(\mathbf{x}) \equiv \varphi(r,\theta,\phi) = A_-(\theta,\phi)\,\frac{e^{-ikr}}{r} + A_+(\theta,\phi)\,\frac{e^{ikr}}{r}\,,$$

where the wave number k is related to the energy of the system $E = k^2/(2m)$.

In practical experimental configurations, the incident wave is rarely chosen to be modelled by a spherical wave, and plane waves are used instead. The modelling (3.2.11) must thus be slightly modified, which is not a big issue as we recall that spherical waves can be well approximated by plane waves at large distances (*i.e.* for $r \to \infty$). Choosing the propagation axis to be the Oz axis, the incident wave is hence considered to be

$$\psi_{\text{inc}}(\mathbf{x}) = e^{ikz}\,, \tag{3.2.12}$$

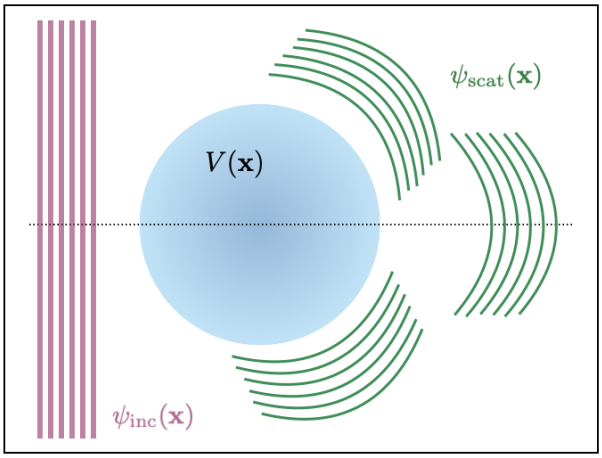


Figure 3.2: Schematic representation of an initial plane wave $\psi_{\text{inc}}(\mathbf{x})$ (purple) that goes through a region in space that is impacted by a potential $V(\mathbf{x})$ (blue). This wave is next scattered in all directions as a spherical wave $\psi_{\text{scat}}(\mathbf{x})$ (green).

in which we consider that the wave propagates toward increasing z values (*i.e.* the exponential is taken to be a positive one). In contrast, far from the scattering centre the outgoing wave has the form of a spherical wave, as in (3.2.11). Equation (3.2.9) can therefore be written at large distances as

$$\psi(\mathbf{x}) = e^{ikz} + f(k,\theta,\phi)\,\frac{e^{ikr}}{r}\,, \tag{3.2.13}$$

which adapts (3.2.11) to an initial state comprising plane waves. In this expression, we have used a mix of Cartesian and spherical coordinates, and we have chosen an outgoing solution for each of the two terms (*i.e.* the wave function includes two positive exponentials). Such a situation is illustrated in figure 3.2. In order to connect this wave function to a cross section, we recall that in quantum mechanics the probability current density $\mathbf{j}(\mathbf{x})$ is defined by

$$\mathbf{j}(\mathbf{x}) = -\frac{i}{2m}\Big[\psi^*(\mathbf{x})\big(\nabla\psi(\mathbf{x})\big) - \big(\nabla\psi^*(\mathbf{x})\big)\psi(\mathbf{x})\Big]\,. \tag{3.2.14}$$

At large distance (*i.e.* for $r \to \infty$), the gradient operator in spherical coordinates simplifies to

$$\nabla \simeq \frac{\partial}{\partial r}\,\mathbf{e}_r\,, \tag{3.2.15}$$

such that we respectively get, for the incident and scattered currents $\mathbf{j}_{\text{inc}}(\mathbf{x})$ and $\mathbf{j}_{\text{scat}}(\mathbf{x})$, the expressions

$$\mathbf{j}_{\text{inc}}(\mathbf{x}) = \frac{k}{m}\,\mathbf{e}_z \qquad \text{and} \qquad \mathbf{j}_{\text{scat}}(\mathbf{x}) = \frac{k}{mr^2}\,\big|f(k,\,\theta,\,\phi)\big|^2\,\mathbf{e}_r\,. \tag{3.2.16}$$

These results are sufficient to deduce the associated cross section from (3.2.4). This equation indeed indicates that the differential cross section corresponds to the ratio of the norm of the scattered current over that of the incident current, normalised to a unit surface. This leads to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r^2\,||\mathbf{j}_{\text{scat}}(\mathbf{x})||}{||\mathbf{j}_{\text{inc}}(\mathbf{x})||} = \big|f(k,\,\theta,\,\phi)\big|^2\,. \tag{3.2.17}$$

Knowledge of the amplitude $f(k,\,\theta,\,\phi)$ therefore allows for a computation of the differential cross section, and then to a comparison of theoretical predictions with experimental data. The expression (3.2.17) is, moreover, applicable to the case of any potential, provided that it decreases with the distance as in (3.2.10). This implies that the potential is almost arbitrary near the origin, that is taken as the scattering centre. The only condition that it must satisfy is

$$V(\mathbf{x}) = \;\mathcal{O}\big(||\mathbf{x}||^{-3/2}\big) \qquad \text{for} \qquad ||\mathbf{x}|| \to 0\,. \tag{3.2.18}$$

Otherwise, the interaction is too strong and the scattering particles may form a bound state (and hence not scatter).

It is generally very hard, and often impossible, to obtain an explicit expression for the amplitude $f(k,\,\theta,\,\phi)$. We can, however, determine a general integral form of it by means of the *Green's function method*, and the expression obtained in this way can be further simplified by means of approximations. This integral form is obtained by first rewriting (3.2.8) as

$$\left[\Delta + k^2\right]\psi(\mathbf{x}) = 2m\,V(\mathbf{x})\,\psi(\mathbf{x})\,, \tag{3.2.19}$$

after using the fact that the energy of the state is that of a free wave, $E = k^2/(2m)$. The general solution to this equation is given by the sum of two components. The first of these consists of the general solution of the associated homogeneous equation,

$$\left[\Delta + k^2\right]\psi(\mathbf{x}) = 0\,, \tag{3.2.20}$$

and the second one consists of a particular solution to (3.2.19). This yields

$$\psi(\mathbf{x}) = \psi_{\text{inc}}(\mathbf{x}) + 2m\int_{\mathbb{R}^3} \mathrm{d}^3x'\; G(\mathbf{x}-\mathbf{x}')\;V(\mathbf{x}')\;\psi(\mathbf{x}')\,. \tag{3.2.21}$$

This solution highlights that we have used the incident wave function $\psi_{\text{inc}}(\mathbf{x})$, *i.e.* a plane wave as shown in (3.2.12), for the general solution of the associated homogeneous equation, and that we have written its particular solution in an integral form relying on the associated Green's function $G(\mathbf{x}-\mathbf{x}')$. The Green's function is defined as the solution to an equation that is similar to (3.2.19), but with a tri-dimensonal δ-type source,

$$\left[\Delta + k^2\right] G(\mathbf{x}-\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,. \tag{3.2.22}$$

This equation can easily be solved in Fourier space. As shown in exercise 3.2, this leads to

$$G(\mathbf{x}-\mathbf{x}') = \frac{i}{4\pi^2||\mathbf{x}-\mathbf{x}'||}\int_{-\infty}^{+\infty}\mathrm{d}q\;\frac{q}{q^2-k^2}\;e^{iq||\mathbf{x}-\mathbf{x}'||}\,. \tag{3.2.23}$$

Exercise 3.2. This exercise is dedicated to the first steps leading to the derivation of the Green's function $G(\mathbf{x}-\mathbf{x}')$ in non-relativistic quantum mechanics. This function is defined as the solution to the equation

$$\left[\Delta + k^2\right] G(\mathbf{x}-\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,.$$

1. Rewrite the above equation in Fourier space to obtain an algebraic equation for the Fourier transform $G(\mathbf{q})$ of the Green's function.
2. Express the Green's function in position space as a tri-dimensional integral over $\mathbf{q}$.
3. Show that integration over the angular variables leads to

$$G(\mathbf{x}-\mathbf{x}') = \frac{i}{4\pi^2||\mathbf{x}-\mathbf{x}'||}\int_{-\infty}^{+\infty}\mathrm{d}q\;\frac{q}{q^2-k^2}\;e^{iq||\mathbf{x}-\mathbf{x}'||}\,.$$

The remaining integral over $q = ||\mathbf{q}||$ can be performed by means of Cauchy's theorem (also known as the residue theorem), once the integration domain is closed in the upper complex plane as illustrated in figure 3.3. Out of the two poles localised in $q = \pm k$, the one at $q = k$ is included in the contour relevant for outgoing waves (left panel of the figure), while the one at $q = -k$ is included in the contour relevant for incoming waves (right panel of the figure). Equation (3.2.22) therefore has two solutions,

$$G_+(\mathbf{x}-\mathbf{x}') = -\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}\,e^{ik||\mathbf{x}-\mathbf{x}'||} \qquad \text{and} \qquad G_-(\mathbf{x}-\mathbf{x}') = -\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}\,e^{-ik||\mathbf{x}-\mathbf{x}'||}\,. \tag{3.2.24}$$

The Green's function $G_+(\mathbf{x}-\mathbf{x}')$ corresponds to an outgoing spherical wave emitted at $\mathbf{x}'$, while the function $G_-(\mathbf{x}-\mathbf{x}')$ corresponds to an incoming spherical wave arriving at $\mathbf{x}'$. Inserting the expression

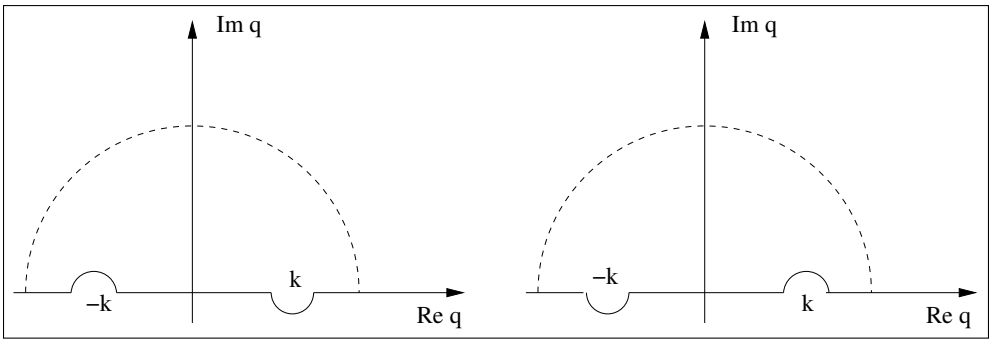


Figure 3.3: Integration contours, in the complex plane, allowing for the calculation of the Green's function in non-relativistic quantum mechanics. These contours correspond to the case of outgoing waves (left) and incoming waves (right).

that we determined for $G_+(\mathbf{x}-\mathbf{x}')$ in the second term of (3.2.21), we can rewrite the Schrödinger equation (3.2.19) in an integral form suitable for scattering studies,

$$\psi(\mathbf{x}) = \psi_{\text{inc}}(\mathbf{x}) - \frac{m}{2\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x' \; \frac{1}{||\mathbf{x}-\mathbf{x}'||}\; e^{ik||\mathbf{x}-\mathbf{x}'||}\; V(\mathbf{x}')\; \psi(\mathbf{x}')\,. \tag{3.2.25}$$

A first possibility to simplify (3.2.25) is to consider its limit at large distances. This condition is often fulfilled as $||\mathbf{x}||$ represents the distance at which the detector lies, *i.e.* the distance at which we are interested in the wave function. On the other hand, the potential, that depends on $\mathbf{x}'$, is only sizable in a small portion of $\mathbb{R}^3$ by virtue of (3.2.10). This equation indeed shows that the potential steeply falls with increasing $||\mathbf{x}'||$ values. Consequently, the integral over the entire space in (3.2.25) is reduced to an integral over a finite region of space in which we can assume that $||\mathbf{x}|| \gg ||\mathbf{x}'||$. This leads to

$$\begin{aligned} k||\mathbf{x}-\mathbf{x}'|| &= k\sqrt{||\mathbf{x}||^2 - 2\mathbf{x}\cdot\mathbf{x}' + ||\mathbf{x}'||^2} \;\simeq\; k||\mathbf{x}|| - \mathbf{k}'\cdot\mathbf{x}'\,, \\ \frac{1}{||\mathbf{x}-\mathbf{x}'||} &= \frac{1}{||\mathbf{x}||}\frac{1}{\left|1-\mathbf{x}\cdot\mathbf{x}'/||\mathbf{x}||^2\right|} \;\simeq\; \frac{1}{||\mathbf{x}||}\,, \end{aligned} \tag{3.2.26}$$

with $\mathbf{k}' = k\,\mathbf{x}/||\mathbf{x}||$ being the wave vector of the scattered particle. Inserting these approximations in (3.2.25), we get the approximate formula

$$\psi(\mathbf{x}) \simeq \psi_{\text{inc}}(\mathbf{x}) - \frac{m}{2\pi\,||\mathbf{x}||}\; e^{ik||\mathbf{x}||}\int_{\mathbb{R}^3} \mathrm{d}^3x'\; e^{-i\mathbf{k}'\cdot\mathbf{x}'}\; V(\mathbf{x}')\; \psi(\mathbf{x}')\,. \tag{3.2.27}$$

Comparing with (3.2.13), we deduce that the amplitude $f(k,\theta,\phi)$ is given by

$$f(k,\theta,\phi) = -\frac{m}{2\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x'\; e^{-i\mathbf{k}'\cdot\mathbf{x}'}\; V(\mathbf{x}')\; \psi(\mathbf{x}')\,. \tag{3.2.28}$$

This expression can be further simplified when the potential is weak. The resulting wave function can then be seen as slightly differing from the incident wave function so that we could replace, to first order, $\psi(\mathbf{x}')$ by $\psi_{\text{inc}}(\mathbf{x}')$ in the integral. This leads to

$$f(k,\theta,\phi) = -\frac{m}{2\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x'\; e^{i\Delta\mathbf{k}'\cdot\mathbf{x}'}\; V(\mathbf{x}') \qquad \text{with} \qquad \Delta\mathbf{k} = \big(k\mathbf{e}_z - \mathbf{k}'\big) = k\big(\mathbf{e}_z - \mathbf{e}_r\big)\,. \tag{3.2.29}$$

In this equation, we have used, for the definition of the quantity $\Delta\mathbf{k}$, mixed Cartesian and spherical coordinates for simplicity. This approximation is called the *Born approximation to first order*. Higher order solutions can be obtained iteratively, the n^{th} order solution being derived by inserting the $(n-1)^{\text{th}}$ order solution in (3.2.27). To first order, the solution (3.2.29) shows that the scattering amplitude $f(k,\theta,\phi)$ is proportional to the Fourier transform $\tilde{V}(\Delta\mathbf{k})$ of the potential,

$$f(k,\theta,\phi) = -4\pi^2 m\; \tilde{V}(\Delta\mathbf{k})\,, \tag{3.2.30}$$

so that we can deduce a compact expression for the the differential cross section from this last relation.

In the Born approximation to first order, the differential cross section associated with scattering processes originating from the effects of a potential $V(\mathbf{x})$ is proportional to the Fourier transform of the potential $\tilde{V}(\Delta\mathbf{k})$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|f(k,\theta,\phi)\right|^2 = 16\pi^4 m^2 \left|\tilde{V}(\Delta\mathbf{k})\right|^2 . \tag{3.2.31}$$

This expression involves the reduced mass m of the system, as well as the vectorial difference $\Delta\mathbf{k}$ between the incident and scattered wave vectors.

The derivation above highlighted an important concept that is also central for scattering computations in QFT, namely that we have free states at very early and late times. We make use of the end of this section to elaborate on this a bit, before moving on with scattering processes in QFT in section 3.2.3.

From now on, we assume that the interaction takes place at the time $t = 0$, that the particles spend very little time in the interaction region, and that the system lies in a time-dependent *scattering state* $|\psi(t)\rangle$. At very early times ($t \to -\infty$) and very late times ($t \to +\infty$), the state of the system must be equivalent to that of a free particle. This consideration allows us to introduce the notion of *asymptotic states* $|\psi_{\text{in}}(t)\rangle$ and $|\psi_{\text{out}}(t)\rangle$, whose time evolution is that of a free particle. However, the time evolution of the three states $|\psi_{\text{in}}(t)\rangle$, $|\psi_{\text{out}}(t)\rangle$ and $|\psi(t)\rangle$ is different. Denoting by $|\psi_{\text{in}}\rangle$, $|\psi_{\text{out}}\rangle$ and $|\psi\rangle$ the three states at $t = 0$, and by $U(t)$ and $U_0(t)$ the time evolution operator of the scattering and free states respectively, we have for the incoming state

$$|\psi_{\text{in}}(t)\rangle = U_0(t)|\psi_{\text{in}}\rangle \quad \text{and} \quad |\psi(t)\rangle = U(t)|\psi\rangle \qquad \text{with} \qquad \lim_{t\to-\infty}\Big(|\psi(t)\rangle - |\psi_{\text{in}}(t)\rangle\Big) = 0\,. \tag{3.2.32}$$

Similarly, the outgoing state satisfies

$$|\psi_{\text{out}}(t)\rangle = U_0(t)|\psi_{\text{out}}\rangle \quad \text{and} \quad |\psi(t)\rangle = U(t)|\psi\rangle \qquad \text{with} \qquad \lim_{t\to+\infty}\Big(|\psi(t)\rangle - |\psi_{\text{out}}(t)\rangle\Big) = 0\,. \tag{3.2.33}$$

We can thus express the scattering state at $t = 0$ ($|\psi\rangle$) as a function of the free states ($|\psi_{\text{in}}\rangle$ and $|\psi_{\text{out}}\rangle$) at the same time,

$$|\psi\rangle = \lim_{t\to-\infty}\Big(U^\dagger(t)U_0(t)|\psi_{\text{in}}\rangle\Big) \equiv \Omega_-|\psi_{\text{in}}\rangle \quad \text{and} \quad |\psi\rangle = \lim_{t\to+\infty}\Big(U^\dagger(t)U_0(t)|\psi_{\text{out}}\rangle\Big) \equiv \Omega_+|\psi_{\text{out}}\rangle\,. \tag{3.2.34}$$

In the above expression, we introduced the so-called *Møller operators* $\Omega_\pm$ intertwining the observables of the free theory with those of the interacting theory, and that carry the name of the Danish physicist Christian Møller (1904 – 1980). These operators additionally allow us to relate the incoming and outgoing asymptotic states,

$$|\psi_{\text{out}}\rangle = \Omega_+^\dagger\Omega_-|\psi_{\text{in}}\rangle \equiv S|\psi_{\text{in}}\rangle\,. \tag{3.2.35}$$

This relation defines the *unitary scattering operator* S, also known as the *S-matrix*. This operator describes a scattering process in which a state $|\psi_{\text{in}}\rangle$ is sent to a state $|\psi_{\text{out}}\rangle$. Furthermore, the S-matrix allows us to calculate the probability that the state $|\psi_{\text{in}}\rangle$ becomes any state $|\phi\rangle$,

$$\mathcal{P}\Big(|\psi_{\text{in}}\rangle \to |\phi\rangle\Big) = \left|\langle\phi|S|\psi_{\text{in}}\rangle\right|^2 . \tag{3.2.36}$$

We can show that the Hamiltonian commutes with the scattering operator S, and that we can recover the expression (3.2.31) of the differential cross section from the S-matrix (and more precisely from its interacting part).

In order to do so, we assume that the system asymptotically lies in the time-independent initial state $|\psi_{\text{in}}(\mathbf{k}_0)\rangle$ of energy $E(k_0) = k_0^2/(2m)$ (with $k_0 = ||\mathbf{k}_0||$), that has not been perturbed (yet) by the potential. Next, we recall that the time evolution of all unperturbed states reads

$$\big|\Psi_{\text{in}}(\mathbf{k},\,t)\big\rangle = e^{-iE(k)t}\,\big|\psi_{\text{in}}(\mathbf{k})\big\rangle\,. \tag{3.2.37}$$

We now need to make the connection with the previous developments in which the incident wave function was given by a plane wave. In order to achieve this, we rely on an expression for the form of the unperturbed state in the position basis that is similar to that of (3.2.12),

$$\Psi_{\text{in}}(\mathbf{x},\,\mathbf{k}_0,\,t) = \langle\mathbf{x}|\Psi_{\text{in}}(\mathbf{k}_0,t)\rangle = \frac{1}{(2\pi)^{3/2}}\,e^{i\left(\mathbf{k}_0\cdot\mathbf{x} - E(k_0)t\right)}\,. \tag{3.2.38}$$

This expression generalises the one of (3.2.12) in the sense that the initial direction has been fixed to an arbitrary vector $\mathbf{k}_0$ instead of $\mathbf{e}_z$. Moreover, its normalisation has been chosen so that this state is normalised to 1, as given by (2.2.42). The transition probability from this state to any outgoing state, together with the associated cross section, is determined in several steps. First, we make use of the fact that the entire set of states $|\Psi_{\text{in}}(\mathbf{k}, t)\rangle$ forms a basis of the Hilbert space. This allows us to expand any state, and in particular the scattering state $|\Psi(t)\rangle$, on the complete set of functions $|\Psi_{\text{in}}(\mathbf{k}, t)\rangle$,

$$\big|\Psi(t)\big\rangle = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\; c(\mathbf{k},\, t)\; e^{-iE(k)t}\; \big|\psi_{\text{in}}(\mathbf{k})\big\rangle\,, \tag{3.2.39}$$

where the Fourier coefficients $c(\mathbf{k},\, t)$ need to be determined. At $t = 0$, *i.e.* before the transition occurs, we have

$$c(\mathbf{k},\, 0) = \delta^{(3)}(\mathbf{k} - \mathbf{k}_0)\,. \tag{3.2.40}$$

The determination of the dependence of the Fourier coefficients on time is achieved by relating them to the potential operator V by means of the time-dependent Schrödinger equation (3.2.5), and by using the orthonormality conditions of the vectors $|\Psi_{\text{in}}(\mathbf{k},\, t)\rangle$. This leads to

$$i\frac{\mathrm{d}}{\mathrm{d}t}\, c(\mathbf{k}',\, t) = \int_{\mathbb{R}^3} \mathrm{d}^3k\; c(\mathbf{k},\, t)\; e^{i\,\omega_{k'k}\,t}\; \langle\psi_{\text{in}}(\mathbf{k}')|V|\psi_{\text{in}}(\mathbf{k})\rangle\,, \tag{3.2.41}$$

as proved in exercise 3.3. The quantity $\langle\psi_{\text{in}}(\mathbf{k}')|V|\psi_{\text{in}}(\mathbf{k})\rangle$ is time-independent and consists of the *transition matrix element* dictating how the potential V enables a transition from a state of energy $E(k)$ and momentum $\mathbf{k}$ to a state of energy $E(k')$ and momentum $\mathbf{k}'$. In this expression, we additionally introduced the *Bohr frequency* defined by

$$\omega_{k'k} = E(k') - E(k)\,. \tag{3.2.42}$$

Exercise 3.3. We assume that a scattering state $|\Psi(t)\rangle$ is defined by the Fourier expansion

$$\big|\Psi(t)\big\rangle = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\; c(\mathbf{k},\, t)\; e^{-iE(k)t}\; \big|\psi_{\text{in}}(\mathbf{k})\big\rangle\,,$$

where $|\psi_{\text{in}}(\mathbf{k})\rangle$ stands for the elements of the basis of unperturbed stationary states. Make use of the Schrödinger equation applied to this scattering state, followed by a projection on the (time-dependent) state $|\Psi_{\text{in}}(\mathbf{k}',\, t)\rangle$, to derive the differential equation driving the time-dependence of the Fourier coefficients $c(\mathbf{k},\, t)$ given by (3.2.41).

In the beginning of this section, we considered a situation in which the potential operator represented a small perturbation from the free theory. This assumption then allowed us to derive the expressions (3.2.28) and (3.2.29) for the scattering amplitude. In order to retrieve these results by means of a formulation in terms of S-matrix elements, we perturbatively expand the potential as a series in a small parameter λ,

$$V = \sum_{i=1}^{\infty} \lambda^i\; V^{(i)}\,. \tag{3.2.43}$$

Since the Fourier coefficients $c(\mathbf{k}',\, t)$ encode the effects of the potential, they can be similarly expanded,

$$c(\mathbf{k}',\, t) = \sum_{i=0}^{\infty} \lambda^i\; c^{(i)}(\mathbf{k}',\, t)\,. \tag{3.2.44}$$

We can insert these two identities in (3.2.41), and next identify the terms independent of λ and those proportional to the first power in λ. By making further use the initial conditions (3.2.40), we get

$$c^{(0)}(\mathbf{k}',\, t) = \delta^{(3)}(\mathbf{k} - \mathbf{k}_0) \qquad \text{and} \qquad c^{(1)}(\mathbf{k}',\, t) = -i\; e^{i\,\omega_{k'k_0}\,t}\; \langle\psi_{\text{in}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle\,. \tag{3.2.45}$$

The $c^{(0)}(\mathbf{k}',\, t)$ coefficient describes the case in which the state does not transition to any other state, whereas the next term in the series $c^{(1)}(\mathbf{k}',\, t)$ allows for the calculation of the transition probability from

a state $|\psi_{\text{in}}(\mathbf{k})\rangle$ to a state $|\psi_{\text{in}}(\mathbf{k}')\rangle$. Integrating this expression between the initial time $t = 0$ and a final time $t = \tau$ (τ being a characteristic time of the interactions), we get

$$c^{(1)}(\mathbf{k}',\,\tau) = \int_0^\tau \mathrm{d}t\; c^{(1)}(\mathbf{k}',\,t) = \frac{1 - e^{i\,\omega_{k'k_0}\,\tau}}{\omega_{k'k_0}}\;\langle\psi_{\text{in}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle\,. \tag{3.2.46}$$

We can now deduce the transition rate per unit of time $\mathrm{d}\mathcal{P}(\mathbf{k}')$ of scattering into a final unperturbed state that we denote $|\psi_{\text{out}}(\mathbf{k}',\,t)\rangle$ and whose wave vector $\mathbf{k}'$ lies in the infinitesimal interval $[\mathbf{k}',\,\mathbf{k}'+\mathrm{d}^3k']$. We get

$$\begin{aligned}\mathrm{d}\mathcal{P}(\mathbf{k}') &= \frac{1}{\tau}\;\left|\langle\psi_{\text{out}}(\mathbf{k}',\,\tau)|\psi(\tau)\rangle\right|^2\mathrm{d}^3k' \\ &= \frac{1}{\tau}\;\left|c^{(1)}(\mathbf{k}',\,\tau)\right|^2\mathrm{d}^3k' \\ &= 2\;\frac{1-\cos\omega_{k'k_0}\tau}{\omega_{k'k_0}^2\,\tau}\;\left|\langle\psi_{\text{out}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle\right|^2\;\mathrm{d}^3k'\,.\end{aligned} \tag{3.2.47}$$

Both the resulting time-independent state $|\psi_{\text{out}}(\mathbf{k}')\rangle$ and initial state $|\psi_{\text{in}}(\mathbf{k}')\rangle$ are defined far from the scattering centre, and hence represent free states. They are thus of identical form. Finally, we can use the expression (3.2.47) to calculate transition rates in the limit $\tau \to \infty$. Taking this limit leads to

$$\mathrm{d}\mathcal{P}(\mathbf{k}') = \pi\;\delta(\omega_{k_0k'})\;\left|\langle\psi_{\text{out}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k_0})\rangle\right|^2\;\mathrm{d}^3k'\,. \tag{3.2.48}$$

The delta function and the definition (3.2.42) of the Bohr frequency imply that energy is conserved. Moreover, (3.2.48) embeds an element squared of the S-matrix by virtue of the definition (3.2.36) applied to a description of transition probability between free states. Equation (3.2.48) can be further simplified after enforcing a change of variable from the radial coordinate $k' = ||\mathbf{k}'||$ to the Bohr frequency $\omega_{k'k_0}$. This allows us to rewrite the integration measure d^3k' as

$$\mathrm{d}^3k' = k'^2\,\mathrm{d}k'\;\mathrm{d}\Omega = 2m\,k'\,\mathrm{d}\omega_{k'k_0}\,\mathrm{d}\Omega\,, \tag{3.2.49}$$

since $\omega_{k'k_0} = E(k') - E(k_0) = k'^2/(2m) - E(k_0)$. Integrating over the frequency, we then get

$$\mathrm{d}\mathcal{P}(\mathbf{k}') = 2\pi\;m\;k_0\;\;\left|\langle\psi_{\text{out}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle\right|^2\;\mathrm{d}\Omega\,. \tag{3.2.50}$$

In order to compute the associated differential cross section (3.2.4), we still have to divide the differential probability (3.2.50) by the incoming flux $F = n_0v_0$ that is given by the product of the initial speed v_0 and the initial number density n_0. The initial speed is simply the ratio of the momentum to the mass, $v_0 = k_0/m$. In order to get an expression for the density of the initial state, we recall the quantum mechanical problem of a particle in a tri-dimensional box of size L, with periodic boundary conditions being applied to the box. The momentum eigenvalues are forced to be a multiple of $(2\pi)/L$, *i.e.* $\mathbf{p} \equiv (p_x, p_y, p_z) = (2n_x\pi/L, 2n_y\pi/L, 2n_z\pi/L)$ with n_x, n_y and n_z being three integers. The volume of momentum space per unit cell in the momentum space is then given by $(2\pi/L)^3$. This equality holds in the continuum limit, so that we get for the number density $n_0 = (2\pi)^{-3}$. This consequently yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{(2\pi)^3 m}{k_0}\;\frac{\mathrm{d}\mathcal{P}(\mathbf{k}')}{\mathrm{d}\Omega} = 16\pi^4\;m^2\;\left|\langle\psi_{\text{out}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle\right|^2\,, \tag{3.2.51}$$

such that we retrieve the results from (3.2.31).

In contrast with (3.2.31), the result is this time written in terms of a matrix element $\langle\psi_{\text{out}}(\mathbf{k}')|V^{(1)}|\psi_{\text{in}}(\mathbf{k}_0)\rangle$, that is by definition proportional to an element of the S-matrix (see (3.2.36)). In the next section, we move on with a description of scattering in a relativistic situation (and therefore in the context of QFT). We make use of the formalism of the S-matrix, that is natural in QFT, and that obviously allows us to recover all previous results as soon as all predictions are evaluated in the non-relativistic limit.

3.2.3 Scattering in field theory

We now consider a relativistic setup in which the number of particles is no longer a conserved quantity, so that QFT has to be used for any practical calculations. As before, the initial state $|i\rangle$ comprises two

particles. It is hence a two-particle state made of two particles a and b of four-momenta p_a^μ and p_b^μ. After scattering occurs, these two particles have annihilated into a final state $|f\rangle$ made of n particles $i_1, \dots, i_n$ of four-momenta p_i^μ with $i = 1, \dots, n$, and we are interested in the computation of the differential cross section $\mathrm{d}\sigma$ given by (3.2.4) using the quantum field theory formalism.

As in the second part of section 3.2.2, we assume that the initial state has been prepared long before the scattering process, and that detection occurs long after the scattering process. The initial and final states therefore correspond to two multiparticle states of the free theory, *i.e.* momentum eigenstates at $t \to \pm\infty$. Like in the non-relativistic case, these states are named *asymptotic states*, and the initial and final state are respectively called the *in state* ($|i\rangle$) and the *out state* ($|f\rangle$). The difference with non-relativistic quantum mechanics is that in QFT, the initial and final states are produced through the action of creation operators $a_{\mathbf{p}}^\dagger$ of definite momentum $\mathbf{p}$ at asymptotically early and late times. In the following, we make use of the properties that we derived in the case of a massive scalar field in section 2.5. However, the relation between the momentum eigenstates and the fields that we use are not limited to real scalar fields. All results derived below therefore safely apply to any kind of fields.

In the example of a $2 \to n$ process, the initial state comprises two particles a and b of momenta $\mathbf{p}_a$ and $\mathbf{p}_b$. The corresponding two-particle state in the free theory is given by

$$|i\rangle = |\mathbf{p}_a\, \mathbf{p}_b\rangle = \sqrt{2E_a}\,\sqrt{2E_b}\; a_{\mathbf{p}_a}^\dagger\, a_{\mathbf{p}_b}^\dagger\, |0\rangle\,, \tag{3.2.52}$$

where the last equality arises from using the definition (2.5.26) twice, in which the energies of the particles a and b are written by E_a and E_b respectively. Strictly speaking, the above description of the initial state is not totally correct. The beams of initial-state particles a and b should instead be represented by wave packets $|\psi_a\rangle$ and $|\psi_b\rangle$ generically given by

$$|\psi_{a,b}\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}\;\; g_{a,b}(\mathbf{p})\; |\mathbf{p}\rangle\,, \tag{3.2.53}$$

where the $g_{a,b}(\mathbf{p})$ are complex functions of the momentum fixed by the initial conditions on the wave packet and satisfying

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}\; \big|g_{a,b}(\mathbf{p})\big|^2 = 1\,. \tag{3.2.54}$$

These functions are centred on the momentum values $\mathbf{p}_a$ and $\mathbf{p}_b$ for the particles a and b respectively, and they steeply fall far from these central values. The width of these wave packets is nevertheless much smaller than any typical experimental resolution so that the structure of the wave packets is never resolved. With this in mind, the initial state can be described by (3.2.52) well, and there is no need to rely on the more complicated expression (3.2.53), that would actually lead to the same results for the cross section. Similarly, as the final state comprises n particles $i_1, \dots, i_n$ of momenta $\mathbf{p}_i$ (with $i = 1, \dots, n$), we have

$$|f\rangle = |\mathbf{p}_1 \,\dots\, \mathbf{p}_n\rangle = \left[\prod_{i=1}^{n} \left(\sqrt{2E_i}\, a_{\mathbf{p}_i}^\dagger\right)\right] |0\rangle\,, \tag{3.2.55}$$

where E_i stands for the energy of the i^{th} particle.

The cross section associated with the process

$$a\, b \;\to\; i_1 \,\dots\, i_n \tag{3.2.56}$$

can be determined from (3.2.4), that shows that it involves the calculation of the incoming flux and that of the differential probability of scattering. In a frame of reference in which the particles are coming from both sides, the incoming flux that is defined by the product of the number density in the beam and the magnitude of the relative velocity $\mathbf{v}_a - \mathbf{v}_b$ of the two particles. It is thus given by $|\mathbf{v}_a - \mathbf{v}_b|$ divided by the total volume V of the interaction region,

$$F = \frac{|\mathbf{v}_a - \mathbf{v}_b|}{V}\,. \tag{3.2.57}$$

This formula is usually evaluated in a specific frame of reference. In a situation in which a beam is fired on a fixed target (like in the case of the Geiger–Marsden experiment), it is common to use the rest frame of the target particles as a standard frame of reference. The expression of the flux then amounts to the ratio of the velocity of the other particle in that frame divided by the volume V. On the other hand, for

collisions of two beams of particles (like at the Large Hadron Collider), we rather use the centre-of-mass frame defined by $\mathbf{p_a} + \mathbf{p_b} = \mathbf{0}$. The expression of the flux is then simplified by using the relation between the velocity $\mathbf{v}$, the momentum $\mathbf{p}$ and the energy E in special relativity, *i.e.* $\mathbf{v} = \mathbf{p}/E$. We get

$$F = \frac{1}{V} \left| \frac{||\mathbf{p}_a||}{E_a} + \frac{||\mathbf{p}_b||}{E_b} \right| = \frac{p_{\text{CM}}}{V} \frac{E_{\text{CM}}}{E_a E_b} \,, \tag{3.2.58}$$

which explicitly involves the centre-of-mass energy E_{CM} and momentum $\mathbf{p}_{\text{CM}}$ defined by $E_a + E_b = E_{\text{CM}}$ and $\mathbf{p}_a = -\mathbf{p}_b = \mathbf{p}_{\text{CM}}$.

In order to get a formula for the differential cross section (3.2.4), we next move on with the derivation of an expression for the differential probability $\mathrm{d}\mathcal{P}$ of the initial state $|i\rangle$ to scatter into the final state $|f\rangle$ in a configuration in which the n final-state particles have momenta lying in a small region $\mathrm{d}^3p_1 \dots \mathrm{d}^3p_n$ of momentum space.

In the previous subsection, we have used the fact that in quantum mechanics, probabilities are given by the squared of an amplitude up to some normalisation factor. Since QFT can be seen as like quantum mechanics after considering an infinite set of harmonic oscillators, a similar definition as the one given in (3.2.36) could be used. However, we recall that the normalisation of a state in QFT is different than that in the non-relativistic case. Factors of the energy indeed need to be introduced to get expressions that satisfy Lorentz invariance, as in (2.5.26) or in the completeness relation (2.5.28). We therefore have

$$\mathrm{d}\mathcal{P} = \frac{\left|\langle f|S|i\rangle\right|^2}{\langle i|i\rangle \, \langle f|f\rangle} \prod_{i=1}^{n} \left[\frac{V}{(2\pi)^3} \, \mathrm{d}^3p_i \right] , \tag{3.2.59}$$

where the last factor defines the relevant region in the space of the final-state particle momenta. The choice for the normalisation factors in (3.2.59) includes several factors of $(2\pi)^3$ (that were also included in (3.2.51)). This ensures that

$$\prod_{i=1}^{n} \left[\frac{V}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3p_i \right] = 1 \,, \tag{3.2.60}$$

such that the sum of all possibilities for the final-state momenta gives 1. As in the derivation of (3.2.51), this equality generalises to the continuous case from the expression for the density of states associated with a particle in a tri-dimensional box of size L with periodic boundary conditions.

The two scalar products $\langle i|i\rangle$ and $\langle f|f\rangle$ and the matrix element $\langle f|S|i\rangle$ appearing in (3.2.59) can be further simplified. First, the definition (2.5.27) of the scalar product of two one-particle states and the commutation relations (2.5.12) imply that

$$\begin{aligned} \langle i|i\rangle &= \langle \mathbf{p}_a|\mathbf{p}_a\rangle \, \langle \mathbf{p}_b|\mathbf{p}_b\rangle \\ &= \left(2E_a \, (2\pi)^3 \, \delta^{(3)}(0)\right) \left(2E_b \, (2\pi)^3 \, \delta^{(3)}(0)\right) \\ &= \left(2E_a \, V\right) \left(2E_b \, V\right) , \end{aligned} \tag{3.2.61}$$

where the last equality highlights the fact that the formally infinite factors of $\delta^{(3)}(0)$ are regularised by the finite volume V allowed for the states. This choice of regulator is well motivated, and it simply originates from the definition of the delta distribution as the Fourier transform of the exponential,

$$\delta^{(3)}(p) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3x \; e^{i\,\mathbf{p}\cdot\mathbf{x}} \quad \Leftrightarrow \quad \delta^{(3)}(0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3x = \frac{V}{(2\pi)^3} \,. \tag{3.2.62}$$

Whereas in (3.2.61), all factors of V are explicitly included, the limit of an infinite volume V will have to be taken at the end of the calculation of the cross section. Moreover, we similarly get for the final state

$$\langle f|f\rangle = \prod_{i=1}^{n} \langle \mathbf{p}_i|\mathbf{p}_i\rangle = \prod_{i=1}^{n} \left(2E_i \, V\right) . \tag{3.2.63}$$

We are thus left with the calculation of the matrix element $\langle f|S|i\rangle$ included in (3.2.59). As already invoked in section 3.2.2, we consider a situation in which the interaction potential is weak so that scattering states can be seen as slightly deviating from free states. Consequently, the scattering operator S is given

by the sum of the identity operator (*i.e.* the case of the free theory) and a small deviation that contains the part of the S-matrix due to interactions,

$$S = 1 + i\,T\,. \tag{3.2.64}$$

The factor of i included in this definition is purely conventional. The operator T is called the *transfer matrix* or the *T-matrix*, and it consists of the interesting part of the S-matrix. We emphasise that even in the case of the free theory ($S = 1$), the matrix element $\langle f|i\rangle$ implies that the two scattering particles have a certain probability to scatter, as well as another probability to miss each other. The relevant matrix element to calculate consequently reads

$$\langle f|\,(S-1)\,|i\rangle = i\langle f|\,T\,|i\rangle\,. \tag{3.2.65}$$

This relation can be rewritten in a more conventional form as

$$i\langle f|\,T\,|i\rangle \equiv i(2\pi)^4\,\delta^{(4)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big)\,\langle f|\,\mathcal{M}\,|i\rangle\,. \tag{3.2.66}$$

This equivalence defines what is traditionally called '*matrix elements*' in the QFT jargon, namely the quantity $\langle f|\,\mathcal{M}\,|i\rangle$. It is analogous to the scattering amplitude appearing in non-relativistic quantum mechanics, appearing for instance in (3.2.41). The rewriting made in (3.2.66) separates the dynamical part of the cross section related to the interaction Hamiltonian (or Lagrangian, as typically used in QFT) from its kinematic part. The dynamics are here embedded in $\mathcal{M}$, while the kinematics are explicitly written and highlight one of the golden rules of physics: *energy and momentum conservation*. The four-dimensional delta function appearing in (3.2.66) reflects that the four-momentum has to be conserved during the scattering process, *i.e.* that

$$p_a^\mu + p_b^\mu = \sum_{i=1}^{n} p_i^\mu\,, \tag{3.2.67}$$

and that $\langle f|\,T\,|i\rangle$ should always contain such a delta function. Otherwise, the corresponding element of the S-matrix vanishes. Finally, we note that (3.2.66) includes a usual conventional factor of $(2\pi)^4$ in the normalisation of the delta function, as in (2.5.12) or (2.5.27).

We are now ready to evaluate the square of the T-matrix element,

$$\big|\langle f|\,T\,|i\rangle\big|^2 = (2\pi)^8\,\delta^{(4)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big)\,\delta^{(4)}(0)\,\big|\langle f|\,\mathcal{M}\,|i\rangle\big|^2\,. \tag{3.2.68}$$

Due to squaring, one of the delta functions ensures that four-momentum is conserved, while the other one is always non-zero and formally infinite. This situation is very similar to that of (3.2.61), and the infinite delta function can be regularised in the same way. Since the delta function is four-dimensional, we must however choose a four-dimensional regulator, that we take to be the finite volume VT in space-time relevant for the scattering process, with T standing for the time during which the whole process occurs. This choice is motivated by

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\; e^{i\,p\cdot x} \quad\Leftrightarrow\quad \delta^{(4)}(0) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x = \frac{VT}{(2\pi)^4}\,. \tag{3.2.69}$$

As for the case of V, the limit of $T \to \infty$ will eventually have to be taken at the end of the calculation of the cross section. We can thus rewrite (3.2.68) as

$$\big|\langle f|\,T\,|i\rangle\big|^2 = (2\pi)^4\,VT\,\delta^{(4)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big)\,\big|\langle f|\,\mathcal{M}\,|i\rangle\big|^2\,. \tag{3.2.70}$$

We now have all the ingredients to calculate the differential cross section $\mathrm{d}\sigma$. We employ the differential probability $\mathrm{d}\mathcal{P}$, that we combine with the results of (3.2.61), (3.2.63) and (3.2.70), as well as with the expression (3.2.57) for the incoming flux. Moreover, the cross section formula should include an overall factor $1/T$ originating from the fact that a cross section is a quantity depicting a probability per unit of time,

$$\mathrm{d}\sigma = \frac{1}{TF}\;\mathrm{d}\mathcal{P}\,. \tag{3.2.71}$$

We therefore get

$$\begin{aligned} \mathrm{d}\sigma &= \frac{1}{T}\,\frac{V}{|\mathbf{v}_a-\mathbf{v}_b|}\,\frac{(2\pi)^4\,VT\,\big|\langle f|\,\mathcal{M}\,|i\rangle\big|^2}{(2E_a\,V)\,(2E_b\,V)}\,\delta^{(4)}\Big(p_a+p_b-\sum_{i=1}^{n}p_i\Big)\,\prod_{i=1}^{n}\left[\frac{V\mathrm{d}^3p_i}{(2\pi)^3}\,\frac{1}{2E_i\,V}\right] \\ &= \frac{1}{|\mathbf{v}_a-\mathbf{v}_b|}\,\frac{(2\pi)^4\,\big|\langle f|\,\mathcal{M}\,|i\rangle\big|^2}{(2E_a)\,(2E_b)}\,\delta^{(4)}\Big(p_a+p_b-\sum_{i=1}^{n}p_i\Big)\,\prod_{i=1}^{n}\left[\frac{\mathrm{d}^3p_i}{(2\pi)^3}\,\frac{1}{2E_i}\right]. \end{aligned} \tag{3.2.72}$$

Amazingly, all the factors of V and T have dropped out, such that the two limits $V\to\infty$ and $T\to\infty$ can be taken trivially. This formula can be rewritten in a more compact way by introducing the n-particle Lorentz-invariant *phase space* $\mathrm{dPS}^{(n)}$.

The integral of the phase space over the final-state momenta is manifestly Lorentz invariant, as it corresponds to integration over invariant three-momenta times a four-dimensional delta function divided by the energy. Furthermore, the matrix element is by construction Lorentz invariant too so that the properties of the cross section under Lorentz transformation only stem from an overall prefactor $1/\mathcal{F}$ defined by,

$$\mathcal{F} = (2E_a)\,(2E_b)\,|\mathbf{v}_a-\mathbf{v}_b|\,. \tag{3.2.73}$$

The derivation of expression (3.2.73) was achieved in the centre-of-mass frame. Not surprisingly $\mathcal{F}$ has consequently the same transformation properties as a cross-sectional area, and is thus invariant under boost transformations along the momentum direction. The form of $\mathcal{F}$ indeed varies under more general Lorentz transformations. We can nevertheless rewrite $\mathcal{F}$ in a Lorentz invariant manner that only relies on Lorentz invariance involving the initial particle masses and the scalar products of their four-momenta,

$$\mathcal{F} = 4\sqrt{(p_a\cdot p_b)^2-m_a^2m_b^2}\,. \tag{3.2.74}$$

The factor $\mathcal{F}$, expressed in this way, is the so-called *Lorentz invariant Møller flux factor*. The expressions (3.2.73) and (3.2.74) further allow for another definition of the relative velocity $\mathbf{v}_{\mathrm{rel}}$ between the initial-state particles, more suitable with respect to special relativity,

$$\mathbf{v}_{\mathrm{rel}} = \frac{1}{p_a\cdot p_b}\,\sqrt{(p_a\cdot p_b)^2-m_a^2m_b^2} = \frac{1}{1-\mathbf{v}_a\cdot\mathbf{v}_b}\,\sqrt{(\mathbf{v}_a-\mathbf{v}_b)^2-(\mathbf{v}_a\times\mathbf{v}_b)^2}\,. \tag{3.2.75}$$

In contrast with the naive definition $v_{\mathrm{rel}}=\mathbf{v}_a-\mathbf{v}_b$, the definition (3.2.75) does not lead to non-sensible results for specific initial-state configurations. For instance, for two initial-state particles traveling at the speed of light towards each other, the naive definition of the relative speed would yield $v_{\mathrm{rel}}=2c$, which is not allowed by the postulates of special relativity. Such a conclusion does not hold with the definition (3.2.75).

In QFT, the differential rate of occurrence that a given $2\to n$ scattering process happens is given by the associated differential cross section $\mathrm{d}\sigma$. For final-state particles with a configuration corresponding to the volume $\mathrm{d}^3p_1\ldots\mathrm{d}^3p_n$ in momentum space, we get

$$\mathrm{d}\sigma = \frac{1}{\mathcal{F}}\,\big|\langle f|\,\mathcal{M}\,|i\rangle\big|^2\,\mathrm{dPS}^{(n)}\,. \tag{3.2.76}$$

This expression involves the Lorentz-invariant n-particle phase space $\mathrm{dPS}^{(n)}$ and Møller flux factor $\mathcal{F}$, that are respectively defined by

$$\begin{aligned} \mathrm{dPS}^{(n)} &= (2\pi)^4\,\delta^{(4)}\Big(p_a+p_b-\sum_{i=1}^{n}p_i\Big)\,\prod_{i=1}^{n}\left[\frac{\mathrm{d}^3p_i}{(2\pi)^3}\,\frac{1}{2E_i}\right], \\ \mathcal{F} &= 4\sqrt{(p_a\cdot p_b)^2-m_a^2m_b^2}\,, \end{aligned} \tag{3.2.77}$$

where p_a^μ and p_b^μ are the four-momenta of the initial-state particles of mass m_a and m_b, and where p_i^μ, $\mathbf{p}_i$ and E_i stand for the four-momentum, the momentum and the energy of the i^{th} final-state particle respectively.

Exercise 3.4. In QFT, the cross section (3.2.76) associated with a generic $2 \to n$ process is expected to be Lorentz invariant. Since matrix elements squared are Lorentz invariant quantities (see the second part of chapter 3), this exercise focuses on the Lorentz invariance of the flux factor and that of the phase space.

1. Demonstrate that in the centre-of-mass frame,

$$\mathcal{F} = (2E_a)\,(2E_b)\,|\mathbf{v}_a - \mathbf{v}_b| = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}\,.$$

   In this expression, the four-momenta of the initial-state particles are given by $p_a^\mu = (E_a, \mathbf{p}_a)$ and $p_b^\mu = (E_b, \mathbf{p}_b)$, and these particles have masses of m_a and m_b respectively.

2. Consider a single final-state particle of mass m and four-momentum $p^\mu = (E, \mathbf{p})$. Show that the corresponding one-body phase space is Lorentz invariant. To this aim, prove that $\mathrm{dPS}^{(1)}$ can be rewritten as an integral over the four-momentum,

$$\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 p}{(2\pi)^4}\,(2\pi)\delta(p^2 - m^2)\;\theta(E)\,,$$

   where $\theta(x)$ stands for the Heaviside function defined by $\theta(x) = 1$ for $x > 0$ and 0 otherwise.

3. Conclude regarding the Lorentz invariance of the cross section.

As mentioned at the beginning of this section, the initial-state particles are organised in beams that should be represented by wave packets like that in (3.2.53). In addition, a similar statement holds for the final-state particles. The structure of the wave packets has, however, no impact on the cross section formula (3.2.76), and we can safely consider the initial-state particles a and b as single particles of definite momenta $\mathbf{p}_a$ and $\mathbf{p}_b$, and the final-state particles $i_1, \ldots, i_n$ as n particles of momenta $\mathbf{p}_i$ (for $i = 1, \ldots, n$). In order to show this, we only focus on the initial state for simplicity. The generalisation to include non-trivial structures for the final state leads to the same conclusions: the formula (3.2.76) is unchanged.

The idealised initial state $|\mathbf{p}_a\,\mathbf{p}_b\rangle$ must be replaced by a state containing two wave packets $|\phi_a\rangle$ and $|\phi_b\rangle$, that are constructed independently at two different locations in space. The corresponding *in state* $|i'\rangle$, the prime indicating the difference with the state $|i\rangle$ of (3.2.52), reads

$$|i'\rangle = |\phi_a\phi_b\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p'_a}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p'_b}{(2\pi)^3}\,\frac{1}{\sqrt{2E'_a}}\,\frac{1}{\sqrt{2E'_b}}\;g_a(\mathbf{p}'_a)\,g_b(\mathbf{p}'_b)\;e^{-i\mathbf{b}\cdot\mathbf{p}'_b}\;|\mathbf{p}'_a\,\mathbf{p}'_b\rangle\,, \tag{3.2.78}$$

where the integrals span all possible values of the momenta $\mathbf{p}'_a$ and $\mathbf{p}'_b$ (as we integrate over them), and that of the energies E'_a and E'_b (that are not independent of the corresponding momenta). This expression shows that the state $|i'\rangle$ consists of a superposition of states $|\mathbf{p}'_a\,\mathbf{p}'_b\rangle$ of the form (3.2.52). The functions $g_a(\mathbf{p})$ and $g_b(\mathbf{p})$ encode the structure of the two wave packets, and we can consider that they are respectively centred on the momenta $\mathbf{p}_a$ and $\mathbf{p}_b$ used in the single-particle case. Moreover, we have chosen the normalisation of the two wave packets such that

$$\langle\phi_a|\phi_a\rangle = \langle\phi_b|\phi_b\rangle = 1\,. \tag{3.2.79}$$

So far, we have adopted an ideal configuration in which the two beams are collinear to each other, and in which they are both aligned on the Oz axis of a reference frame in momentum space. In such a case, they are said to feature a *relative impact parameter* $\mathbf{b} = \mathbf{0}$. In practical situations, however, the two beams can be transversally misaligned. Such a configuration must thus be included in the cross section formula, which is what is modelled by the exponential factor $e^{-i\mathbf{b}\cdot\mathbf{p}'_b}$ in (3.2.78). In the latter formula, we have opted to align the Oz axis in momentum space onto beam a, that we have thus chosen as a reference. Consequently, we only need a single two-dimensional vector $\mathbf{b}$ to describes the transverse spatial misalignment of the second beam b relative to the reference beam a. The cross section therefore includes a two-dimensional integral over the impact parameter $\mathbf{b}$ to account for all possible misalignments.

It is finally worth mentioning that in the normalisation conventions of (3.2.78), the incoming Møller flux factor is equal to 1, as we have appropriately normalised the state $|i'\rangle$. The differential cross section

is thus given by

$$\mathrm{d}\sigma = \prod_{i=1}^{n} \left[\frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \right] \int_{\mathbf{R}^2} \mathrm{d}^2 b \, \left| \langle f | \, T \, | i' \rangle \right|^2 . \tag{3.2.80}$$

The two wave packets are, however, well localised in momentum space. Consequently, we can evaluate any factor of the cross section that is a smooth function of the momenta at momentum values $\mathbf{p}'_a \simeq \mathbf{p}_a$ and $\mathbf{p}'_b \simeq \mathbf{p}_b$. Together with the normalisation conditions of the two wave packets, all extra integrals appearing in the cross section formula (relative to (3.2.76)) can be evaluated, and we can show that we retrieve formula (3.2.72). The explicit derivation is left as an exercise (exercise 3.5).

Exercise 3.5. Consider the differential cross section (3.2.80),

$$\mathrm{d}\sigma = \prod_{i=1}^{n} \left[\frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \right] \int_{\mathbf{R}^2} \mathrm{d}^2 b \, \left| \langle f | \, T \, | i' \rangle \right|^2 ,$$

depicting a scattering process in which two wave packets ϕ_a and ϕ_b annihilate to produce a final state $|f\rangle$ of n particles $i_1, \ldots, i_n$ of momenta $\mathbf{p}_i$ and energies E_i. The initial state $|i'\rangle = |\phi_a \phi_b\rangle$ involved in the cross section is defined by (3.2.78).

1. Rewrite the cross section formula above as a 14-dimensional integral over the initial state and T-matrix elements of the form $\langle f | \, T \, | i \rangle$ (without the prime on the initial state). These integrals consist of a single two-dimensional integral over the components of the impact parameter, two three-dimensional integrals over the $|\mathbf{p}'_a\rangle$ states making the first wave packet, and two three-dimensional integrals over the $|\mathbf{p}'_b\rangle$ states making the second wave packet.
2. Rewrite the squared T-matrix element in terms of $|\langle f | \, \mathcal{M} \, | i \rangle|^2$, which yields two four-dimensional delta functions.
3. Show that the evaluation of the two-dimensional integral over the impact parameter $\mathbf{b}$ leads to an additional two-dimensional delta function.
4. Evaluate six of the remaining integrals by means of six of the delta functions.
5. Make use of the normalisation condition of the wave packets, and the fact that they are well centred on the momenta $\mathbf{p}_a$ and $\mathbf{p}_b$, to evaluate the six remaining integrals.
6. Show that the formula obtained in this way is identical to (3.2.76).

3.2.4 Spinless particle scattering in quantum electrodynamics

We consider as a simple example of scattering processes in QFT the case of a $2 \to 2$ process in which two particles a and b of four-momentum $p_a^\mu = (E_a, \mathbf{p}_a)$ and $p_b^\mu = (E_b, \mathbf{p}_b)$ annihilate into two particles i_1 and i_2 of four momenta $p_1^\mu = (E_1, \mathbf{p}_1)$ and $p_2^\mu = (E_2, \mathbf{p}_2)$,

$$a(p_a) \quad b(p_b) \quad \to \quad i_1(p_1) \quad i_2(p_2) \, . \tag{3.2.81}$$

This example allows us to highlight several important features of calculations in QFT. First of all, $2 \to 2$ processes are standard in many high-energy physics experiments so that studying them generically is useful. Next, scattering processes generally involve the propagation of fields from the source that generates them. The associated notion of *propagator* is therefore crucial, as propagators appear everywhere, and this example allows us to introduce these objects smoothly. Finally, we take the example of electromagnetism that we review in its covariant way.

Results associated with $2 \to 2$ scattering processes are often expressed in terms of the *Mandelstam variables* s, t and u defined by [25]

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2 \, , \qquad t = (p_a - p_1)^2 = (p_b - p_2)^2 \, , \qquad u = (p_a - p_2)^2 = (p_b - p_1)^2 \, . \tag{3.2.82}$$

These variables were introduced by Stanley Mandelstam (1928 – 2016), and they represent a Lorentz-invariant way to encode the energy, the momentum, and the angular orientation of all particles in a $2 \to 2$ scattering process. They moreover satisfy

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2 \, , \tag{3.2.83}$$

where m_a and m_b stand for the masses of the initial-state particles a and b, and m_1 and m_2 stand for the masses of the final-state particles i_1 and i_2. Only two of the Mandelstam variables are thus independent.

Exercise 3.6. We consider the generic scattering process

$$a(p_a) \quad b(p_b) \quad \to \quad i_1(p_1) \quad i_2(p_2)\,,$$

in which two particles a and b of four-momenta $p_a = (E_a, \mathbf{p}_a)$ and $p_b = (E_b, \mathbf{p}_b)$ annihilate into two particles i_1 and i_2 of four-momenta $p_1 = (E_1, \mathbf{p}_1)$ and $p_2 = (E_2, \mathbf{p}_2)$. The four particle masses are moreover denoted by m_a, m_b, m_1 and m_2.

1. Rewrite all four-momenta in terms of the Mandelstam variable s, the four masses, and the polar angle θ defined as the angle between the momenta $\mathbf{p}_a$ and $\mathbf{p}_1$.
2. Rewrite $\cos\theta$ in terms of the Mandelstam variables and the particle masses.
3. Show that when the initial-state particles are massless, the phase space can be expressed as

$$\mathrm{dPS}^{(2)} = \frac{\mathrm{d}t}{8\,\pi\,s}\,.$$

We begin our calculation of the cross section associated with a $2 \to 2$ process by adapting the formula (3.2.76) to the case considered. In the centre-of-mass frame, we have

$$E_a + E_b = E_1 + E_2 \equiv E_{\mathrm{CM}}\,, \qquad \mathbf{p}_a = -\mathbf{p}_b \qquad \text{and} \qquad \mathbf{p}_1 = -\mathbf{p}_2\,. \tag{3.2.84}$$

With this in mind, we can calculate the flux factor and the phase space entering the cross section. These quantities being Lorentz invariant, they can be calculated in any frame. In the centre-of-mass frame, we get, from (3.2.73),

$$\mathcal{F} = (2E_a)\,(2E_b)\,|\mathbf{v}_a - \mathbf{v}_b| = 4\,|E_b\,\mathbf{p}_a - E_a\,\mathbf{p}_b| = 4\;E_{\mathrm{CM}}\,p_i\,, \tag{3.2.85}$$

where the norm of the initial-state momenta is given by $p_i = ||\mathbf{p}_a|| = ||\mathbf{p}_b||$. On the other hand, the two-body phase space $\mathrm{dPS}^{(2)}$ is given by (3.2.77), which reduces to

$$\mathrm{dPS}^{(2)} = (2\pi)^4\,\delta^{(4)}\big(p_a + p_b - p_1 - p_2\big)\;\frac{\mathrm{d}^3p_1}{(2\pi)^3}\,\frac{\mathrm{d}^3p_2}{(2\pi)^3}\,\frac{1}{2E_1}\,\frac{1}{2E_2}\,. \tag{3.2.86}$$

The integral over p_2 can be evaluated by means of three of the delta functions, which gives

$$\mathrm{dPS}^{(2)} = (2\pi)\,\delta\big(E_a + E_b - E_1 - E_2\big)\;\frac{\mathrm{d}^3p_1}{(2\pi)^3}\,\frac{1}{2E_1}\,\frac{1}{2E_2}\,, \tag{3.2.87}$$

with the condition $\mathbf{p}_2 = \mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 = -\mathbf{p}_1$ being enforced. The last delta function allows for the integration over the norm of the momentum. This leads to

$$\mathrm{dPS}^{(2)} = \frac{1}{16\pi^2}\;\mathrm{d}\Omega\;\mathrm{d}p_f\;\frac{p_f^2}{E_1E_2}\;\delta\big(E_a + E_b - E_1 - E_2\big)\,. \tag{3.2.88}$$

In this expression, $\mathrm{d}\Omega$ stands for the angular variables associated with $\mathbf{p}_1$, and we introduced the norm of the final-state momenta, $p_f = ||\mathbf{p}_1|| = ||\mathbf{p}_2||$. The definition of the energy according to Einstein implies that

$$E_1 = \sqrt{m_1^2 + p_f^2} \qquad \text{and} \qquad E_2 = \sqrt{m_2^2 + p_f^2}\,. \tag{3.2.89}$$

This allows us to rewrite the argument inside the last delta function of the phase space expression (3.2.88) in the form

$$\delta\big(E_a + E_b - E_1 - E_2\big) = \frac{1}{p_f}\;\frac{E_1E_2}{E_{\mathrm{CM}}}\;\delta\big(p_f - \dots\big)\,, \tag{3.2.90}$$

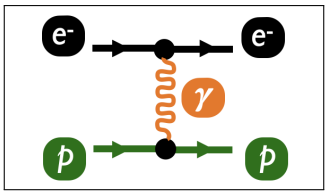


Figure 3.4: Feynman diagram associated with the process $e^- p \to e^- p$ in a case where all interactions are of an electromagnetic nature. The initial state is shown on the left part of the diagram, and the final state on its right part. The electron beam (black) and the proton target (green) can be seen as exchanging a photon (orange), and the dots represent two electromagnetic interaction vertices (one for the electron and one for the proton).

and then perform the integral over p_f. This yields

$$\mathrm{dPS}^{(2)} = \frac{1}{16\pi^2}\,\mathrm{d}\Omega\,\frac{p_f}{E_{\mathrm{CM}}}\,\theta\big(m_1+m_2-E_{\mathrm{CM}}\big)\,, \tag{3.2.91}$$

where the Heaviside function shows that the energy of the initial state must be at least as large as the masses of the two final-state particles for the process to occur. Plugging everything into (3.2.76), we get an expression for the cross section that is valid in the centre-of-mass frame, $(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\mathrm{CM}}$,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2 E_{\mathrm{CM}}^2}\,\frac{p_f}{p_i}\,\big|\mathcal{M}\big|^2\,\theta\big(m_1+m_2-E_{\mathrm{CM}}\big)\,, \tag{3.2.92}$$

where we have introduced a (standard) shortcut notation $\big|\mathcal{M}\big|^2$ for the squared matrix element.

We now apply formula (3.2.92) to the case of a specific scattering process. We consider that an electron beam is fired on a fixed target made of protons, and we restrict our calculation to a context in which all interactions of protons and electrons are driven by electromagnetism. In QFT, electromagnetic interactions can be seen as the exchange of photons between the interacting particles, as detailed in the next chapters. Moreover, protons do not interact directly with electrons. The process considered must thus be seen as an electron from the beam exchanging a photon with one of the protons in the target. This is pictorially represented in figure 3.4. This figure consists of what we call a *Feynman diagram*, and it has a definite meaning that is further discussed in the next sections. In practice, Feynman diagrams allow for a pictorial representation of a scattering process, that can next be translated into an equation corresponding to the associated matrix element $\mathcal{M}$.

In the following, we calculate the cross section associated with the process $e^- p \to e^- p$ in the non-relativistic limit so that our findings can be compared with the results of section 3.2.2. Moreover, any electron and proton spin effects are neglected for the reason that we have only handled scalar fields representing spinless particles so far. This derivation is thus a purely academic exercise as protons and electrons are spin-1/2 particles. This is, however, sufficient for our current purpose. In our non-relativistic $2 \to 2$ setup, the centre-of-mass energy is given approximately by the proton mass m_p. This originates from the facts that the proton mass is about 2000 times greater than the electron mass m_e, and that in a non-relativistic limit all particles are close to being at rest. We thus have

$$E_{\mathrm{CM}} \simeq m_p\,, \tag{3.2.93}$$

and the proton rest frame can be used as the centre-of-mass frame to first order. Moreover, the scattering is elastic as all final-state particles are identical to the initial-state ones. We therefore have

$$p_i = p_f\,, \tag{3.2.94}$$

and the cross section (3.2.92) is simplified to

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2 m_p^2}\,\big|\mathcal{M}\big|^2\,. \tag{3.2.95}$$

We are thus left with the calculation of the squared matrix element $|\mathcal{M}|^2$. Calculating this exactly relies on *Feynman rules*, which is a topic covered in the next sections of this chapter. We nevertheless

have enough information to do it in an approximate and non-rigorous way, that is in fact sufficient to show that the non-relativistic limit of QFT predictions matches the expectation of non-relativistic quantum mechanics. In order to do so, we start from the Lagrangian $\mathcal{L}$ describing the electromagnetic interactions of spinless particles of charge q and mass m with photons (that mediate electromagnetic interactions). It is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^\dagger\big(\Box + m^2\big)\phi + ieqA^\mu(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger) + \mathcal{O}(e^2)\,. \tag{3.2.96}$$

The first term in this Lagrangian (in which all dependencies on the space-time coordinates are omitted for clarity) describes the free photon dynamics and involves the field strength tensor

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\,. \tag{3.2.97}$$

The second term is the Klein-Gordon Lagrangian driving the dynamics of scalar particles of mass m, and the last term consists of their electromagnetic interaction with photons that have a strength proportional to the unit of charge e. We now provide additional details on the different terms of this Lagrangian.

In the covariant treatment of electromagnetism, the four-vector $A^\mu(x)$ is identified with the electromagnetic potential $A^\mu(x) = \big(V(x),\, \mathbf{A}(x)\big)$, in which $V(x)$ is the associated scalar potential and $\mathbf{A}(x)$ the vector potential, and the field strength tensor $F^{\mu\nu}(x)$ is written in terms of the components of the magnetic field $\mathbf{B}(x) = \big(B_x(x),\, B_y(x),\, B_z(x)\big)$ and electric field $\mathbf{E}(x) = \big(E_x(x),\, E_y(x),\, E_z(x)\big)$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x(x) & -E_y(x) & -E_z(x) \\ E_x(x) & 0 & -B_z(x) & B_y(x) \\ E_y(x) & B_z(x) & 0 & -B_x(x) \\ E_z(x) & -B_y(x) & B_x(x) & 0 \end{pmatrix}. \tag{3.2.98}$$

With this definition, Maxwell's equations, that get their names from James Maxwell (1831 – 1879), have the simple forms

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x) \qquad \text{and} \qquad \partial^\mu F^{\nu\rho}(x) + \partial^\nu F^{\rho\nu}(x) + \partial^\rho F^{\nu\mu}(x) = 0\,, \tag{3.2.99}$$

where the source term $j^\mu(x) = \big(\rho(x),\, \mathbf{j}(x)\big)$ combines an external current $\mathbf{j}(x)$ and charge density $\rho(x)$. It can be shown that any field $F^{\mu\nu}(x)$ that consists of a solution to these equations can always be written as (3.2.97).

The first set of equations in (3.2.99) can alternatively be obtained by means of the Euler-Lagrange equations (2.4.10), once we start from Maxwell's Lagrangian,

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A^\mu j_\mu\,. \tag{3.2.100}$$

This consists of our starting point, and we refer to exercise 3.7 for the full derivation of Maxwell's equations in terms of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. More explicitly, making use of Euler-Lagrange equations and the Lagrangian (3.2.100) gives

$$\Box A^\nu(x) - \partial^\nu\partial_\mu A^\mu(x) = j^\nu(x)\,. \tag{3.2.101}$$

At this stage, we did not specify the electromagnetic current $j^\mu(x)$, that is arbitrary, and our results are thus general. For example, in the Lagrangian (3.2.96) this current depends on another field ϕ,

$$j^\mu(x) = ieq\Big(\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)\Big) + \mathcal{O}(e^2)\,. \tag{3.2.102}$$

To further simplify (3.2.101), we enforce the *Lorenz gauge* condition,

$$\partial_\mu A^\mu(x) = 0\,. \tag{3.2.103}$$

Such a condition, that is also Lorentz invariant, is named after Ludvig Lorenz (1829 – 1891), and it allows for a (partial) elimination of the spin-0 degree of freedom embedded in Maxwell's equation. This is necessary due to the fact that the electromagnetic field $A^\mu(x)$ is made up of spin-1 particles called *photons*. However, the corresponding $(1/2, 1/2)$ representation of the Lorentz algebra includes four degrees of freedom, which leads to the $0 \oplus 1$ representation of $\mathfrak{so}(3)$. The $(1/2, 1/2)$ representation can therefore

be used to describe both spin-0 and spin-1 particles (see section 1.3.2), and the former must be prevented from propagating. Additional technical details are left for chapter **??**.

Maxwell's equations (3.2.101) read, in the Lorenz gauge, as

$$\Box A^\nu(x) = j^\nu(x)\,. \tag{3.2.104}$$

The form of this set of four equations is very similar to that of the Klein-Gordon equation (2.5.3), at least in the massless case and once we account for the external current. This is not surprising as the photon field A_μ is a vector field containing four components ($\mu = 0, 1, 2, 3$), and each component field is a scalar field satisfying the Klein-Gordon equation. Those four fields are however not arbitrary, and they must depend on each other. The exact form of this dependence can be derived from the polarisation state of the photon, as further detailed in chapter **??**.

Exercise 3.7. Consider Maxwell's Lagrangian

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A^\mu j_\mu\,.$$

1. Demonstrate that this Lagrangian yields Maxwell's equations by means of the Euler-Lagrange equations.
2. Rewrite these equations in terms of the components of the field strength tensor $F^{\mu\nu}$, and retrieve the usual form of Maxwell's equations involving the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.
3. Derive the equations satisfied by the scalar potential V and the vector potential $\mathbf{A}$ in the Lorenz gauge.
4. Show that both the Lorenz gauge condition and Maxwell's equations are preserved by the transformation
$$A^\mu \to A^\mu + \partial^\mu f\,,$$
with f being any harmonic scalar function (*i.e.* a function such that $\Box f = 0$).

The set of four equations (3.2.104) has the formal solution

$$A^\mu(x) = \Pi_A\, j^\mu(x)\,, \tag{3.2.105}$$

where the operator $\Pi_A \equiv \Box^{-1}$ is the inverse of the d'Alembert operator. Such an operator Π_A is generically called a *propagator*, as it expresses how to define the field $A^\mu(x)$ once it propagates from the source $j^\mu(x)$. In order to solve this expression and to derive an (approximate) expression for the photon propagator, we recall some basic properties of the d'Alembert operator $\Box$ in Fourier space.

The d'Alembert operator satisfies the following important equality,

$$\Box^n f(x) = \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \Box^n f(k) e^{ik\cdot x} = \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \big(-k^2\big)^n f(k) e^{ik\cdot x}\,, \tag{3.2.106}$$

where the integral is performed over the full Minkowski space-time $\mathbb{R}^{1,3}$, and where n is an integer. This property is all that is needed to define the action of the d'Alembert operator on a field such as the one defined in (2.5.20). Such a field is indeed defined from a tri-dimensional integral over an integrand containing an exponential factor $e^{ik\cdot x}$. The action of the d'Alembert operator therefore leads to a multiplication of the integrand by a factor of $-k^2$. Such a procedure in which the d'Alembert operator is replaced by some four-momentum squared is standard in QFT calculations, and it can be applied to the case of the photon propagator in (3.2.105). The photon propagator must thus include a factor of

$$-\frac{1}{k^2} \tag{3.2.107}$$

in Fourier space, where k^μ stands for the photon four-momentum. This expression may seem surprising as photons are massless particles, so that $k^2 = 0$, implying that the above factor is ill-defined. It is, however,

important to mention that Feynman diagrams often involve *virtual particles* for which the associated equations of motion are not satisfied. Consequently, $k^2 \neq 0$ and (3.2.107) can be used. Moreover, this expression is not the full expression of the photon propagator (that is derived in chapter **??**). However, equation (3.2.107) includes enough information to calculate the cross section (3.2.95) for the process $e^- p \to e^- p$ in an approximate manner.

Exercise 3.8. We consider a setup in which the electromagnetic potentials are generated by the effect of a single positive charge localised at the origin. The corresponding external current can be written, after using an appropriate system of units in which the vacuum permittivity $\varepsilon_0 = 1$, as

$$j^\mu(x) = \left(e\delta^{(3)}(\mathbf{x}),\, \mathbf{0}\right).$$

1. Show that the photon propagator should include in this case a factor $1/||\mathbf{k}||^2$.
2. Solve Maxwell's equations (in the Lorenz gauge) by rewriting the delta function appearing in the expression of the current as a tri-dimensional integral of an exponential.
3. Show that this allows us to retrieve the usual expression for the Coulomb potential.

We now go back to the Lagrangian (3.2.96) and briefly focus on the second term. As mentioned above, this terms consists of the Klein-Gordon Lagrangian dictating the dynamics of the field ϕ and the scalar particles that it describes. As for the photon field A^μ, we can extract from it an expression for the scalar field propagator. This is, however, not necessary for the present exercise so that we do not focus more on this term. The final term is more interesting as it dictates how a pair of ϕ particles interact with the photon. After accounting for the expression (2.5.20) for the field ϕ, we can evaluate the action of the space-time derivative operator ∂_μ on the field: it consists of a multiplication by the four-momentum. Note that particles are mostly at rest in the non-relativistic limit, *i.e.* $p^\mu \simeq (m, \mathbf{0})$. Consequently, we can solely focus on time derivatives, which leads to

$$ieqA^\mu(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) \simeq -2meqA^0 \phi^\dagger \phi\,. \tag{3.2.108}$$

This expression shows that the strength of the electromagnetic interaction in the non-relativistic limit is proportional to the mass and the charge of the interacting particles. Plugging all ingredients together, we obtain, for the matrix element associated with the $e^- p \to e^- p$ process,

$$\mathcal{M} \sim (-2em_p)\, \frac{1}{||\mathbf{k}||^2}\, (2em_e)\,. \tag{3.2.109}$$

The first factor corresponds to the piece describing the interaction of the proton with a photon and is given by (3.2.108), whereas the last factor corresponds to that of the electron and is given by the same equation. The second factor consists of the photon propagator, for which we have used the results of exercise 3.8, as we are interested in the propagation of the electromagnetic field once it has been generated by the proton at rest. Finally, we can derive an expression for the cross section (3.2.95),

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} \sim \frac{e^4 m_e^2}{4\pi^2}\, \frac{1}{||\mathbf{k}||^4} = 16\pi^2 m_e^2 \left|\tilde{V}(\mathbf{k})\right|^2, \tag{3.2.110}$$

where in the last equality we inserted the Fourier transform $\tilde{V}(\mathbf{k})$ of the Coulomb potential $V(\mathbf{x})$,

$$V(\mathbf{x}) = \frac{e^2}{4\pi||\mathbf{x}||} \qquad \Leftrightarrow \qquad \tilde{V}(\mathbf{k}) = \frac{e^2}{8\pi^3||\mathbf{k}||^2}\,. \tag{3.2.111}$$

Whereas the calculation that we performed in this section is non-rigorous in the sense that we made use of approximate expressions for the Feynman rules associated with the interactions and the photon propagator, all ingredients to be included in the exact calculation were already present. Moreover, the cross section (3.2.110) agrees with the non-relativistic quantum mechanical expectation (3.2.31).

In contrast to what we have done in the current subsection, in the rest of these notes we will be more rigorous. In particular, the notion of matrix elements is further characterised in section 3.4, in which a more precise derivation is done from first principles, *i.e.* from the definition (2.5.20) of a quantum field.

3.3 Decay rates

In section 3.2, we studied scattering processes in which two initial-state particles annihilate into some final state made up of n particles, and we discussed the derivation of a formula for a quantity known as a (differential) cross section. This quantity is measurable experimentally, and it can hence be used to confront quantum field theoretical predictions with experimental data. Other quantities that are relevant to test a given quantum field theory are those associated with processes involving a single initial-state particle a that *decays* into some final state comprising of n particles $i_1, \dots, i_n$,

$$a \to i_1 \dots i_n \,. \tag{3.3.1}$$

Similar to the differential cross section $\mathrm{d}\sigma$ introduced in the context of $2 \to n$ scattering processes, we can define the so-called *differential decay rate* $\mathrm{d}\Gamma$ describing the dynamics of the decay process. Decays usually happen over a finite and short time so that they can be considered as $1 \to n$ scattering processes. The formalism detailed in section 3.2 can thus be identically applied to $1 \to n$ processes, and the formula (3.2.76), taken together with (3.2.77), can be adapted.

For a process, such as that in (3.3.1), in which a particle a of four-momentum $p_a^\mu = (E_a, \mathbf{p}_a)$ and mass m_a decays into a set of n particles $i_1, \dots, i_n$ of four-momenta $p_i^\mu = (E_i, \mathbf{p}_i)$ and masses m_i, the differential decay rate is given by

$$\mathrm{d}\Gamma = \frac{1}{2E_a} \left| \langle f | \, \mathcal{M} \, | i \rangle \right|^2 \mathrm{dPS}^{(n)} \,. \tag{3.3.2}$$

Here $|i\rangle$ and $|f\rangle$ generically denote the initial state and the final state of the process. As in the case of $2 \to n$ scattering, the phase space is defined by (3.2.77), *i.e.*

$$\mathrm{dPS}^{(n)} = (2\pi)^4 \, \delta^{(4)}\Big(p_a - \sum_{i=1}^{n} p_i\Big) \prod_{i=1}^{n} \left[\frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \right] . \tag{3.3.3}$$

The factor of $1/E_a$ included in (3.3.2) shows that if the particle a is relativistic, *i.e.* if its energy E_a is much larger than its mass m_a, the decay will be slower. This is the effect of time dilatation, that explains, for instance, why we can see cosmic muons on Earth before they decay. In the case of a two-body decay (*i.e.* a $1 \to 2$ process), we can perform calculations in the centre-of-mass frame. In this case, the formula (3.3.2) simplifies to

$$\big(\mathrm{d}\Gamma\big)_{\mathrm{CM}} = \frac{p_{\mathrm{CM}}}{32\,\pi^2\,m_a^2} \left|\mathcal{M}\right|^2 \mathrm{d}\Omega = \frac{p_{\mathrm{CM}}}{32\,\pi^2\,m_a^2} \left|\mathcal{M}\right|^2 \mathrm{d}\phi \, \mathrm{d}\cos\theta \,, \tag{3.3.4}$$

where the magnitude of the final-state particle momenta in the centre-of-mass frame is given by $p_{\mathrm{CM}} = ||\mathbf{p}_1|| = ||\mathbf{p}_2||$. Moreover, we used the notation $\mathcal{M}$ for the matrix element $\langle f | \, \mathcal{M} \, | i \rangle$, and $\mathrm{d}\Omega$ corresponds to an infinitesimal solid angle in momentum space in the direction (θ, ϕ), relative to the scattering centre.

Exercise 3.9. Demonstrate that the differential decay rate $\mathrm{d}\Gamma$ associated with the decay process

$$a \to i_1 \, i_2$$

can be written, in the centre-of-mass frame, as

$$\big(\mathrm{d}\Gamma\big)_{\mathrm{CM}} = \frac{p_{\mathrm{CM}}}{32\,\pi^2\,m_a^2} \left|\mathcal{M}\right|^2 \mathrm{d}\phi \, \mathrm{d}\cos\theta \,.$$

In our notation, $p_{\mathrm{CM}} = ||\mathbf{p}_1|| = ||\mathbf{p}_2||$, m_a stands for the mass of particle a, and the origin of the reference frame corresponds to the origin of the rest frame of particle a.

After phase space integration, we obtain the so-called *partial decay width* $\Gamma(a \to i_1 \dots i_n)$ of the associated process. We can further define the *total decay width* Γ_a of the particle a as the sum of the

partial decay widths of all possible decay processes for the particle a. The total width allows us to evaluate the time evolution of the density $N(t)$ of particles of type a in a certain setup,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\Gamma_a N(t) \qquad \Leftrightarrow \qquad N(t) = N_0\, e^{-\Gamma_a t}\,, \tag{3.3.5}$$

with N_0 denoting the density at the initial time. Moreover, its is connected to the lifetime τ_a of the particle a,

$$\tau_a = \frac{1}{\Gamma_a}\,. \tag{3.3.6}$$

3.4 The LSZ reduction formula

In section 3.2, we derived the expression (3.2.66) for the elements of the S-matrix,

$$\langle f|\,(S-1)\,|i\rangle = i(2\pi)^4\,\delta^{(4)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big)\,\langle f|\,\mathcal{M}\,|i\rangle\,. \tag{3.4.1}$$

However, we did not provide any specific means to calculate the matrix element $\mathcal{M} \equiv \langle f|\,\mathcal{M}\,|i\rangle$. In order to do so, the first step, that we tackle in this section, consists of the derivation of the so-called *Lehmann-Symanzik-Zimmermann (LSZ) reduction formula* that relates S-matrix elements to quantum fields. This formula further leads to the concept of *Feynman rules*, which provides diagrammatic expansions allowing for a more or less easy derivation of the matrix element associated to any scattering process.

Consider the generic $2 \to n$ process

$$a(p_a)\; b(p_b) \;\to\; i_1(p_1)\;\ldots\; i_n(p_n)\,, \tag{3.4.2}$$

where p_a^μ and p_b^μ stand for the four-momenta of the initial-state particles a and b, and where $p_1^\mu, \ldots, p_n^\mu$ are the four-momenta of the n final-state particles $i_1, \ldots, i_n$. The initial state $|i\rangle$ appearing on the left-hand side of (3.4.1) is an asymptotic state obtained from the action, at $t \to -\infty$, of the creation operators $a^\dagger_{\mathbf{p}_a}(t)$ and $a^\dagger_{\mathbf{p}_b}(t)$ on the vacuum state of the interacting theory $|\Omega\rangle$. We recall that $|\Omega\rangle$ consists of a state in which there is no particle, and we emphasise that the state $|\Omega\rangle$ differs from the ground state $|0\rangle$ of the free theory introduced in section 2.5. However, we consider a theoretical framework in which the Hamiltonian H of the interacting theory is given by the sum of the free Hamiltonian H_0, and an interaction Hamiltonian H_{int}. The two ground states $|0\rangle$ and $|\Omega\rangle$ must thus overlap to some extent, *i.e.* $\langle\Omega|0\rangle \neq 0$.

In addition, creation and annihilation operators at a time t are in general different from creation and annihilation operators at another time t'. The action of the interacting part of the Hamiltonian indeed leads to operator mixing through time evolution. These points are clarified in section 3.5, in which we show how the mapping of the two vacua $|\Omega\rangle$ and $|0\rangle$ automatically involves the interaction Lagrangian of the theory. For the moment, it is only sufficient to consider that the creation and annihilation operators $a^\dagger_{\mathbf{p}}$ and $a_{\mathbf{p}}$ act in the usual way on the vacuum $|\Omega\rangle$.

One-particle states $|\mathbf{p}\rangle$ are normalised as in (2.5.26), with an energy-dependent prefactor. This leads to the definition of the $|i\rangle$ state as

$$|i\rangle = \lim_{\tau\to-\infty}\Big(\sqrt{2\omega_{\mathbf{p}_a}}\;\sqrt{2\omega_{\mathbf{p}_b}}\;a^\dagger_{\mathbf{p}_a}(\tau)\,a^\dagger_{\mathbf{p}_b}(\tau)\,|\Omega\rangle\Big)\,, \tag{3.4.3}$$

where $\omega_{\mathbf{p}} = \sqrt{||\mathbf{p}||^2 + m^2}$ stands for the energy of the state $|\mathbf{p}\rangle$ (representing some particle of mass m). Similarly, the final state $|f\rangle$ is given by

$$|f\rangle = \lim_{\tau\to+\infty}\Big(\prod_{i=1}^{n}\sqrt{2\omega_{\mathbf{p}_i}}\;a^\dagger_{\mathbf{p}_i}(\tau)\,|\Omega\rangle\Big)\,. \tag{3.4.4}$$

Assuming that the two states $|i\rangle$ and $|f\rangle$ are different, then only the T-matrix component of the S-matrix contributes. We can write in this case

$$\langle f|\,S\,|i\rangle = \lim_{\tau\to\infty}\Big(\sqrt{2^{n+2}\;\omega_{\mathbf{p}_a}\,\omega_{\mathbf{p}_b}\,\omega_{\mathbf{p}_1}\,\ldots\,\omega_{\mathbf{p}_n}}\;\langle\Omega|a_{\mathbf{p}_n}(\tau)\,\ldots\,a_{\mathbf{p}_1}(\tau)\,a^\dagger_{\mathbf{p}_a}(-\tau)\,a^\dagger_{\mathbf{p}_b}(-\tau)\,|\Omega\rangle\Big)\,. \tag{3.4.5}$$

In this expression, the relative ordering of all annihilation operators among themselves and that of all creation operators among themselves are both irrelevant, as they commute. We can relate (3.4.5) to the quantum field ϕ defined in (2.5.20) by introducing the *time-ordering operator* $T\{\dots\}$. The action of this operator on a product of operators orders them regardless of their commutation properties in such a way that operators that act at later times are always on the left of operators that act at earlier times. We can therefore rewrite (3.4.5) as

$$\langle f|\,S\,|i\rangle = \sqrt{2^{n+2}\;\omega_{\mathbf{p}_a}\,\omega_{\mathbf{p}_b}\,\omega_{\mathbf{p}_1}\,\dots\,\omega_{\mathbf{p}_n}}\;\lim_{\tau\to\infty}\left(\langle\Omega|\;T\{a_{\mathbf{p}_n}(\tau)\,\dots\,a_{\mathbf{p}_1}(\tau)\,a^\dagger_{\mathbf{p}_a}(-\tau)\,a^\dagger_{\mathbf{p}_b}(-\tau)\}\;|\Omega\rangle\right). \tag{3.4.6}$$

In (3.4.5), all creation operators acting at $t\to-\infty$ were already placed to the right of all annihilation operators acting at $t\to\infty$. The product of creation and annihilation operators appearing inside the element of the S-matrix considered is therefore already time-ordered, and (3.4.5) and (3.4.6) are equivalent. We can then make use of the properties of the time-ordering operator to include additional vanishing terms in the matrix element,

$$\begin{aligned}\langle f|\,S\,|i\rangle = \sqrt{2^{n+2}\;\omega_{\mathbf{p}_a}\,\omega_{\mathbf{p}_b}\,\omega_{\mathbf{p}_1}\,\dots\,\omega_{\mathbf{p}_n}}\;\lim_{\tau\to\infty}\Big(\Big\langle\Omega\Big|\;T\Big\{\;&\big(a_{\mathbf{p}_n}(\tau)-a_{\mathbf{p}_n}(-\tau)\big)\,\dots\\ &\times\big(a_{\mathbf{p}_1}(\tau)-a_{\mathbf{p}_1}(-\tau)\big)\,\big(a^\dagger_{\mathbf{p}_a}(-\tau)-a^\dagger_{\mathbf{p}_a}(\tau)\big)\,\big(a^\dagger_{\mathbf{p}_b}(-\tau)-a^\dagger_{\mathbf{p}_b}(\tau)\big)\Big\}\;\Big|\Omega\Big\rangle\Big).\end{aligned} \tag{3.4.7}$$

The equality with the previous expression is ensured by the fact that the time ordering leads, for all extra terms, to a product of operators featuring either a creation operator on the left, or an annihilation operator on the right. Since by definition we have

$$a|\Omega\rangle = \langle\Omega|a^\dagger = 0\,, \tag{3.4.8}$$

any product of operators included in (3.4.7) but not included in (3.4.6) annihilates the vacuum. In order to rewrite (3.4.7) in terms of a quantum field ϕ, we make use of the property

$$i\int_{\mathbb{R}^{1,3}}\mathrm{d}^4x\;e^{ip\cdot x}\;\left(\Box+m^2\right)\;\phi(x) = \sqrt{2\omega_{\mathbf{p}}}\;\lim_{\tau\to\infty}\big(a_{\mathbf{p}}(\tau)-a_{\mathbf{p}}(-\tau)\big)\,, \tag{3.4.9}$$

for any four-vector $p^\mu=(\omega_{\mathbf{p}},\mathbf{p})$ such that $p^2=m^2$. In order to demonstrate this property, we recall that the field $\phi(x)$ is assumed to vanish at the spatial boundaries of the integration domain. Integration by parts over the spatial coordinates of the left-hand side of (3.4.9) then leads to

$$\begin{aligned}i\int_{\mathbb{R}^{1,3}}\mathrm{d}^4x\;e^{ip\cdot x}\;\left(\Box+m^2\right)\;\phi(x) &= i\int_{\mathbb{R}^{1,3}}\mathrm{d}^4x\;e^{ip\cdot x}\;\left(\frac{\partial^2}{\partial t^2}+\omega^2_{\mathbf{p}}\right)\;\phi(x)\\ &= \int_{\mathbb{R}}\mathrm{d}t\frac{\partial}{\partial t}\left[e^{i\omega_{\mathbf{p}}t}\int_{\mathbb{R}^3}\mathrm{d}^3x\;e^{-i\mathbf{p}\cdot\mathbf{x}}\;\left(i\frac{\partial}{\partial t}+\omega_{\mathbf{p}}\right)\;\phi(x)\right]\\ &= \left[e^{i\omega_{\mathbf{p}}t}\int_{\mathbb{R}^3}\mathrm{d}^3x\;e^{-i\mathbf{p}\cdot\mathbf{x}}\;\left(i\frac{\partial}{\partial t}+\omega_{\mathbf{p}}\right)\;\phi(x)\right]_{-\infty}^{+\infty},\end{aligned} \tag{3.4.10}$$

where for the second equality we re-organised the different factors of the integrand to show that it could be written as a total time-derivative. Inserting the definition (2.5.20) of the field $\phi(x)$, we can simplify the integral in the squared bracket,

$$\begin{aligned}&\left[e^{i\omega_{\mathbf{p}}t}\int_{\mathbb{R}^3}\mathrm{d}^3x\;e^{-i\mathbf{p}\cdot\mathbf{x}}\;\left(i\frac{\partial}{\partial t}+\omega_{\mathbf{p}}\right)\;\phi(x)\right]_{-\infty}^{+\infty}\\ &\quad=\left[e^{i\omega_{\mathbf{p}}t}\int_{\mathbb{R}^3}\mathrm{d}^3k\int_{\mathbb{R}^3}\frac{\mathrm{d}^3x}{(2\pi)^3}\left(\frac{\omega_{\mathbf{k}}+\omega_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{k}}}}a_{\mathbf{k}}(t)e^{-i\omega_{\mathbf{k}}t}e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}+\frac{\omega_{\mathbf{p}}-\omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}a^\dagger_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}t}e^{-i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}\right)\right]_{-\infty}^{+\infty}\\ &\quad=\left[e^{i\omega_{\mathbf{p}}t}\sqrt{2\omega_{\mathbf{p}}}\;a_{\mathbf{p}}(t)\;e^{-i\omega_{\mathbf{p}}t}\right]_{-\infty}^{+\infty},\end{aligned} \tag{3.4.11}$$

using firstly the fact that at infinitely early and late time the ladder operators can be considered as time-independent, and secondly the definition of the Fourier transform of the delta function given in (2.5.23). This last result proves (3.4.9). Consequently, we can rewrite (3.4.7) entirely in terms of quantum fields.

The LSZ reduction formula shows that the calculation of an S-matrix element amounts to evaluating a time-ordered product of fields on which some $\Box + m^2$ operators act, and perform the Fourier transform of the result,

$$\begin{aligned}\langle f|\, S\,|i\rangle = &\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_n\; e^{ip_n\cdot x_n}\;\left(\Box_n + m^2\right)\right]\ldots\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_1\; e^{ip_1\cdot x_1}\;\left(\Box_1 + m^2\right)\right]\\ &\times\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_a\; e^{-ip_a\cdot x_a}\;\left(\Box_a + m^2\right)\right]\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_b\; e^{-ip_b\cdot x_b}\;\left(\Box_b + m^2\right)\right]\\ &\times\left\langle\Omega\,\middle|\, T\Big\{\;\phi(x_n)\ldots\phi(x_1)\phi(x_a)\phi(x_b)\;\Big\}\,\middle|\,\Omega\right\rangle,\end{aligned} \tag{3.4.12}$$

with

$$\Box_k = \frac{\partial}{\partial x_k^\mu}\,\frac{\partial}{\partial x_{k\mu}}\,. \tag{3.4.13}$$

In the case where all fields would be free fields, the quantity $(\Box + m^2)\phi(x) = (-p^2 + m^2)\phi(x)$ consists of the equations of motion associated with field $\phi(x)$, and is thus equal to zero. Consequently, any matrix element should be zero. However, propagators involve denominators given by the inverse of the $(\Box + m^2)$ operator, *i.e.* factors of $1/(p^2 - m^2)$. The propagator factors can thus cancel some of these zeros (instead of yielding a divergence). In explicit calculations relevant for a scattering process $i \to f$, propagators associated with all particles in the initial and final state of interest appear. The LSZ reduction formula therefore plays the role of isolating the asymptotic states of interest for the specific $i \to f$ process from the numerous terms typically included in a products of fields. This is achieved practically by multiplying by a zero (obtained through the equations of motion of the different particles) any term that does not correspond to the initial state or the final state considered.

We remind ourselves that the initial state is defined as the set of particles created from the vacuum at an infinitely early time, and that the final state is similarly defined to be the set of particles annihilating into the vacuum at an infinitely late time. On the other hand, we emphasise the role of the time ordering operator. The latter implies that the initial state must be created before the final state annihilates. In addition, it additionally guarantees that any temporary state that would emerge at an intermediate time is created before it annihilates.

One of the strengths of the LSZ formula is that it applies regardless of the form of the fields. The only requirement is that the field $\phi(x)$ is an operator capable of creating and annihilating particles at asymptotic times. It does not matter whether this operator is related to elementary particles, bound states of elementary particles, or that it could even be any generic operator. The LSZ reduction formula is in this case generalised as

$$\begin{aligned}\langle f|\, S\,|i\rangle = &\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_n\; e^{ip_n\cdot x_n}\;\left(\Box_n + m^2\right)\right]\ldots\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x_1\; e^{-ip_1\cdot x_1}\;\left(\Box_1 + m^2\right)\right]\\ &\times\left\langle\Omega\,\middle|\, T\Big\{\;\mathcal{O}(x_n)\ldots\mathcal{O}(x_1)\;\Big\}\,\middle|\,\Omega\right\rangle,\end{aligned} \tag{3.4.14}$$

where each of the $\mathcal{O}_i$ operators can create and annihilate one-particle states,

$$\langle\mathbf{p}|\mathcal{O}(x)|\Omega\rangle \propto e^{ip\cdot x}\,. \tag{3.4.15}$$

These operators can thus be defined as a linear combination of products of annihilation and creation operators (as in quantum mechanics), or they can be constructed as functions of fields (as is typical in QFT). The non-zero overlap with the one-particle states $|\mathbf{p}\rangle$ is what guarantees the generation of a pole in the matrix element so that the associated $\Box + m^2$ zero of the LSZ formula is cancelled.

In the next section, we exemplify the calculation of a time-ordered product of fields in QFT by considering particle propagation in the free theory, before moving on with the description of a method allowing for the calculation of time-ordered products of fields in an interacting theory. This second step is handled in the next two sections.

3.5 The Feynman propagator

In the previous section, we derived the LSZ formula (3.4.14) as a first step to gaining insights on how to calculate the squared matrix elements that were entering the formulas (3.2.76) of a cross section and (3.3.2) of a decay width. In particular, we noted that formula (3.4.14) involved the vacuum state $|\Omega\rangle$ of the interacting theory, and not the vacuum $|0\rangle$ of the free theory. Moreover, expressions for the fields in the free theory, that are denoted $\phi_0(x)$ from now on, can be different from expressions for the fields in the interacting theory, as they depend on ladder operators that evolve differently with time. Time-evolution is indeed driven by the free Hamiltonian H_0 in the free theory, and by the full Hamiltonian $H = H_0 + H_{\text{int}}$ in the interacting theory. We therefore have to express not only the vacuum state $|\Omega\rangle$ of the interacting theory in terms of the free theory fields $\phi_0(x)$ and the vacuum state $|0\rangle$ that we know how to manipulate, but also the fields in the interacting theory in terms of the same quantities.

In order to elaborate on the last points from a concrete example, we perform, as a first calculation of a time-ordered product of fields, the calculation of a product of free fields $\phi_0(x)$. Such a field is given by (2.5.20), and reads

$$\phi_0(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}}\, e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}\, e^{ip\cdot x} \right). \tag{3.5.1}$$

Here, as usual in the case of free fields $\omega_{\mathbf{p}} = \sqrt{m^2 + ||\mathbf{p}||^2}$, and the annihilation and creation operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ are time-independent. Moreover, as we consider the free theory the vacuum is $|0\rangle$. In this setup, the simplest time-ordered product of fields that can be calculated involves only two occurrences of the free field $\phi_0(x)$. It is given by

$$G_F(x_2 - x_1) \equiv \langle 0 \,|\, T\{\phi_0(x_2)\phi_0(x_1)\} \,|\, 0 \rangle . \tag{3.5.2}$$

This quantity $G_F(x_2 - x_1)$ corresponds to the amplitude dictating how a particle propagates in space-time from x_1 to x_2. In order to evaluate this expression, we begin with the calculation of the associated quantity $G(x_2 - x_1)$ that does not include any time-ordering operator,

$$G(x_2 - x_1) \equiv \langle 0 \,|\, \phi_0(x_2)\phi_0(x_1) \,|\, 0 \rangle . \tag{3.5.3}$$

The calculation of $G(x_2 - x_1)$ is achieved by replacing the free field $\phi_0(x)$ by its expression (3.5.1), and by making use of the properties of the creation and annihilation operators when they act on the vacuum. As an annihilation operators annihilates the vacuum ket ($a|0\rangle = 0$), and a creation operator annihilates the corresponding bra ($\langle 0|a^\dagger = 0$), only terms with a creation operator on the right and an annihilation operator on the left survive,

$$\begin{aligned} G(x_2 - x_1) &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_2}}} \frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}} \langle 0 \,|\, a_{\mathbf{p}_2}\, a^\dagger_{\mathbf{p}_1} \,|\, 0 \rangle\, e^{-i(p_2\cdot x_2 - p_1 \cdot x_1)} \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_1}} e^{-ip_1\cdot(x_2 - x_1)} , \end{aligned} \tag{3.5.4}$$

where the second equality arises from the commutation relations (2.5.12). We can now easily calculate $G_F(x_2 - x_1)$ by making sure that the operator which acts first lies on the right of that which acts next. Therefore, $T\{\phi_0(x_2)\phi_0(x_1)\}$ yields two terms, one for $t_1 > t_2$ and one for $t_1 < t_2$. Introducing the time difference $\Delta t = t_2 - t_1$, we get

$$\begin{aligned} G_F(x_2 - x_1) &= \langle 0 \,|\, \phi_0(x_2)\phi_0(x_1) \,|\, 0 \rangle\, \theta(\Delta t) + \langle 0 \,|\, \phi_0(x_1)\phi_0(x_2) \,|\, 0 \rangle\, \theta(-\Delta t) \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-ip\cdot(x_2 - x_1)}\, \theta(\Delta t) + e^{-ip\cdot(x_1 - x_2)}\, \theta(-\Delta t) \right) \\ &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}_2 - \mathbf{x}_1)} \left(e^{-i\omega_{\mathbf{p}}\Delta t}\, \theta(\Delta t) + e^{i\omega_{\mathbf{p}}\Delta t}\, \theta(-\Delta t) \right), \end{aligned} \tag{3.5.5}$$

where we recall that the Heaviside function $\theta(x)$ is defined by $\theta(x) = 1$ for $x > 0$ and 0 otherwise. The two terms correspond to what is usually called the *retarded* and *advanced* propagators. Their sum can be simplified by relying on the property

$$e^{-i\omega_{\mathbf{p}}\Delta t}\, \theta(\Delta t) + e^{i\omega_{\mathbf{p}}\Delta t}\, \theta(-\Delta t) = \lim_{\varepsilon \to 0} \frac{-\omega_{\mathbf{p}}}{\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\omega^2 - \omega^2_{\mathbf{p}} + i\varepsilon}\, e^{i\omega\Delta t} , \tag{3.5.6}$$

and inserting it in (3.5.5). This allows us to retrieve the usual expression for the propagator of a scalar particle.

The quantity $G_F(x-y)$ is called the *Feynman propagator* for a particle described by the Klein-Gordon equation. It is Lorentz-invariant and reads

$$G_F(x-y) \equiv \langle 0 \,|\, T\{\phi_0(x)\phi_0(y)\} \,|\, 0\rangle = \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip\cdot(x-y)}\, \frac{i}{p^2-m^2+i\varepsilon}\,. \tag{3.5.7}$$

This shows that the Feynman propagator is the Fourier transform of the function

$$\tilde{G}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}\,, \tag{3.5.8}$$

which is crucial in the formalism of Feynman diagrams. As will be shown in section 3.6, such an expression is attached to any internal line in a Feynman diagram, that represents the propagation of a *virtual particle* (for which the related equations of motion are not satisfied).

We can notice that the Feynman propagator is similar to the Green's function that we derived in section 3.2.2. This is not a coincidence as the Feynman propagator is actually a Green's function. It therefore satisfies an equation that is similar to (3.2.19),

$$\left(\Box + m^2\right) G_F(x-y) = -i\delta^{(4)}(x-y)\,, \tag{3.5.9}$$

with specific boundary conditions.

The Feynman propagator has a pole at $p^2 = m^2$, and it hence has the potential to cancel one of the p^2-m^2 factors of the LSZ reduction formula. Moreover, we have omitted the limit of $\varepsilon \to 0$ in expressions (3.5.7) and (3.5.8). However, this limit has to be considered implicitly. The $+i\varepsilon$ term that appears in the denominator is indeed a necessary ingredient of the pole prescription used for the regularisation of the contour integrals. It is related to the factors of Heaviside functions $\theta(\pm\Delta t)$ included in the expression of $G_F(x-y)$ when it is written as a product of fields without the time-ordering operator. Furthermore, the integral over the four-momentum ($\int \mathrm{d}^4p$) has now to be evaluated over the full four-momentum space. The propagating particle can thus be *off-shell*, with $p^0 \neq \sqrt{||\mathbf{p}||^2+m^2}$ as p^0 is now an independent integration variable, and it can thus be virtual.

Exercise 3.10. This exercise relies on the application of Cauchy's theorem to derive an expression for the Feynman propagator $G_F(x-y)$ as a Fourier transform.

1. Make use of the partial fraction decomposition to rewrite $1/(\omega^2 - E^2 + i\varepsilon)$ as a sum of two fractions containing a single pole in ω, for ε being an infinitesimal quantity.
2. By means of Cauchy's theorem, demonstrate that for $E > 0$ the following equalities are satisfied:

$$\int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\omega - E + i\varepsilon}\, e^{i\omega t} = -2i\pi e^{iEt}\,\theta(-t) + \mathcal{O}(\varepsilon) \qquad \text{and} \qquad \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\omega + E - i\varepsilon}\, e^{i\omega t} = 2i\pi e^{iEt}\,\theta(t) + \mathcal{O}(\varepsilon)\,.$$

3. Deduce that

$$e^{-i\omega_{\mathbf{p}}\Delta t}\,\theta(\Delta t) + e^{i\omega_{\mathbf{p}}\Delta t}\,\theta(-\Delta t) = \lim_{\varepsilon\to 0} \frac{-\omega_{\mathbf{p}}}{\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\omega^2 - \omega_{\mathbf{p}}^2 + i\varepsilon}\, e^{i\omega\Delta t}\,.$$

4. Make use of these results to express the Feynman propagator as

$$G_F(x_1 - x_0) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{1,3}} \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip\cdot(x_1-x_0)}\, \frac{i}{p^2-m^2+i\varepsilon}\,.$$

5. Show that the Feynman propagator is a Green's function for the Klein-Gordon equation.

We have so far derived the Feynman propagator in the free theory, as we started from the associated vacuum $|0\rangle$. This, however, corresponds to a theory in which individual particles live in isolated modes, oblivious to any other particles either of the same species or of different species. On the contrary, calculations relevant for scattering processes, and the possibility of making observations, require us to consider an interacting theory. We shall now investigate how the Feynman propagator (3.5.7) changes in an interacting theory described by the Hamiltonian $H = H_0 + H_{\text{int}}$, where H_0 stands for the free Hamiltonian and H_{int} for the interaction one. Denoting by $|\Omega\rangle$ the vacuum state in the interacting theory, we must thus calculate

$$\langle \Omega \,|\, T\{\phi(x_2)\phi(x_1)\} \,|\, \Omega \rangle \,. \tag{3.5.10}$$

In comparison with (3.5.2), the two occurrences of the free field $\phi_0(x)$ have been replaced by their counterparts $\phi(x)$ of the interacting theory, and the vacuum state of the free theory $|0\rangle$ has been replaced by $|\Omega\rangle$.

In section 2.2.3 we introduced the Schrödinger picture in which time evolution applies to states, while operators are instead time-independent and therefore fixed. In addition, we also defined the Heisenberg picture in which case time evolution applied to operators, while states are in contrast time-independent and fixed. Finally, we also mentioned the existence of a third possibility, namely the *interaction picture*. In this case time evolution is split into two components, one of them being applied to states and the other to operators. This hybrid possibility is the one that we consider here: we enforce that the time dependence of operators is governed by H_0, whereas that of states is governed by H_{int}. A Schrödinger-picture state $|\psi_{\mathcal{S}}(t)\rangle$ and an interaction-picture state $|\psi_{\mathcal{I}}(t)\rangle$ are thus related through the free evolution operator $U_0(t, t_0)$

$$|\,\psi_{\mathcal{I}}(t)\,\rangle = U_0^\dagger(t, t_0)\;|\,\psi_{\mathcal{S}}(t)\,\rangle = e^{iH_0(t-t_0)}\;|\,\psi_{\mathcal{S}}(t)\,\rangle \,. \tag{3.5.11}$$

The two states coincide at $t = t_0$, and we recall that the free evolution operator $U_0(t, t_0)$ satisfies a differential equation of the form (2.2.73)

$$i\frac{\partial}{\partial t}U_0(t, t_0) = H_0\; U_0(t, t_0) \,. \tag{3.5.12}$$

Conversely, the evolution over time of an operator $\mathcal{A}_{\mathcal{I}}(t)$ in the interaction picture is given by,

$$\mathcal{A}_{\mathcal{I}}(t) = U_0^\dagger(t, t_0)\;\mathcal{A}_{\mathcal{S}}\;U_0(t, t_0) = e^{iH_0(t-t_0)}\;\mathcal{A}_{\mathcal{S}}\;e^{-iH_0(t-t_0)} \,, \tag{3.5.13}$$

and hence involves the time-independent operator $\mathcal{A}_{\mathcal{S}}$ of the Schrödinger picture. This last relation applies in particular to the interaction Hamiltonian H_{int}, so that we can define the time-dependent interaction-picture Hamiltonian $H_{\mathcal{I}}(t)$,

$$H_{\mathcal{I}}(t) = U_0^\dagger(t, t_0)\;H_{\text{int}}\;U_0(t, t_0) = e^{iH_0(t-t_0)}\;H_{\text{int}}\;e^{-iH_0(t-t_0)} \,. \tag{3.5.14}$$

Inserting the expressions (3.5.11) and (3.5.14) in the Schrödinger equation (2.2.56), we get

$$i\frac{\partial}{\partial t}|\psi_{\mathcal{I}}(t)\rangle = H_{\mathcal{I}}(t)\;|\psi_{\mathcal{I}}(t)\rangle \,. \tag{3.5.15}$$

This confirms that the time evolution of the interaction-picture states are solely driven by the interaction Hamiltonian. We now have the necessary ingredients to move on with the calculation of (3.5.10).

Exercise 3.11. Demonstrate that the evolution of an interaction-picture state $|\psi_{\mathcal{I}}(t)\rangle$ is driven by the differential equation

$$i\frac{\partial}{\partial t}|\psi_{\mathcal{I}}(t)\rangle = H_{\mathcal{I}}(t)\;|\psi_{\mathcal{I}}(t)\rangle \,,$$

in which $H_{\mathcal{I}}(t)$ stands for the interaction-picture Hamiltonian.

At infinitely early times, the system lies in an asymptotic state whose expression can be computed from the free theory. There must therefore exist a reference time t_0 where the free field $\phi_0(x)$ is matched onto the field $\phi(x)$. We make use of this time t_0 as the starting point of the time evolution of the fields. In the interacting theory, the evolution of the field $\phi(x)$ from t_0 is given by (2.2.73),

$$\phi(t, \mathbf{x}) = U_{\text{full}}^\dagger(t, t_0)\;\phi(t_0, \mathbf{x})\;U_{\text{full}}(t, t_0) \,, \tag{3.5.16}$$

where we made explicit the position and time dependence of the fields for clarity. The field $\phi(t_0, \mathbf{x})$ is time-independent (as t is fixed to t_0) and consists of the Schrödinger-picture field, whereas the field $\phi(x) = \phi(t, \mathbf{x})$ depends on time and is thus the Heisenberg-picture field. We have removed the $\mathcal{H}$ and $\mathcal{S}$ subscripts as the Heisenberg/Schrödinger picture version of the field can be directly deduced from their time-dependence. In addition, the evolution operator $U_{\text{full}}(t, t_0)$ satisfies the usual differential equation (2.2.57) of an evolution operator, that reads

$$i\frac{\partial}{\partial t}U_{\text{full}}(t, t_0) = H\ U_{\text{full}}(t, t_0)\,. \tag{3.5.17}$$

We define the *interaction-picture field* $\phi_{\mathcal{I}}(x)$ from (3.5.13). Such a field corresponds to a time evolution of the Schrödinger-picture field $\phi(t_0, \mathbf{x})$ from t_0 to t, that is it is solely driven by the free Hamiltonian H_0,

$$\begin{aligned}\phi_{\mathcal{I}}(x) &= e^{iH_0(t-t_0)}\ \phi(t_0, \mathbf{x})\ e^{-iH_0(t-t_0)} \\ &= U_0^\dagger(t, t_0)\ \phi(t_0, \mathbf{x})\ U_0(t, t_0)\,.\end{aligned} \tag{3.5.18}$$

Since at $t = t_0$ the two fields are matched by the definition of the reference time t_0 (*i.e.* $\phi(t_0, \mathbf{x}) = \phi_0(t_0, \mathbf{x})$), the interaction-picture field is equal to the free field $\phi_I(x) = \phi_0(x)$. Consequently, we will always use the notation $\phi_0(x)$ for the interaction-picture field in the following.

After inverting (3.5.18) and inserting it in the relation (3.5.16), we can relate the field $\phi(x)$ to the free field $\phi_0(x)$ at any given time t,

$$\begin{aligned}\phi(x) &= U_{\text{full}}^\dagger(t, t_0)\ U_0(t, t_0)\ \phi_0(x)\ U_0^\dagger(t, t_0)\ U_{\text{full}}(t, t_0) \\ &\equiv U_{\mathcal{I}}^\dagger(t, t_0)\ \phi_0(x)\ U_{\mathcal{I}}(t, t_0)\,.\end{aligned} \tag{3.5.19}$$

In the last part of this equation, we have introduced a new unitary operator, the *interaction-picture time-evolution operator* $U_{\mathcal{I}}(t, t_0)$,

$$U_{\mathcal{I}}(t, t_0) = U_0^\dagger(t, t_0)\ U_{\text{full}}(t, t_0)\,. \tag{3.5.20}$$

When acting on a state, this operator evolves it forward in time from t_0 to t on the basis of the full Hamiltonian H, and then backward in time from t to t_0 on the basis of the free Hamiltonian H_0. The operator $U_{\mathcal{I}}(t, t_0)$ can be expressed entirely in terms of the interaction-picture Hamiltonian $H_{\mathcal{I}}(t)$ by working out its time dependence. This leads to the differential equation,

$$i\frac{\partial}{\partial t}U_{\mathcal{I}}(t, t_0) = H_{\mathcal{I}}(t)\ U_{\mathcal{I}}(t, t_0)\,, \tag{3.5.21}$$

with initial conditions given by

$$U_{\mathcal{I}}(t_0, t_0) = 1\,, \tag{3.5.22}$$

and where the interaction-picture Hamiltonian is defined in (3.5.14). Equation (3.5.21) shows that there is a way to express the evolution operator entirely in terms of the time-dependent free fields $\phi_0(x)$. In order to determine this expression, we first note that there is no reason that the interaction-picture Hamiltonian $H_{\mathcal{I}}(t)$ should commute with itself when evaluated at different times. The solution to the differential equation (3.5.21) must subsequently be written as a Dyson series, similarly to (2.2.68). The properties of the time-ordering operator then further allows us to write this series in a compact exponential form,

$$U_{\mathcal{I}}(t,\, t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t'\, H_{\mathcal{I}}(t')\right]\right\}. \tag{3.5.23}$$

The detailed proof that this can be done is the subject of exercise 3.12. In addition, we can also show through a computation of the time-derivative of (3.5.23) that the time dependence of the operator $U_{\mathcal{I}}(t, t_0)$ matches the differential equation (3.5.21),

$$\begin{aligned}\frac{\partial}{\partial t}T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t'\, H_{\mathcal{I}}(t')\right]\right\} &= -i\ T\left\{H_{\mathcal{I}}(t)\ \exp\left[-i\int_{t_0}^{t} \mathrm{d}t'\, H_{\mathcal{I}}(t')\right]\right\} \\ &= -iH_{\mathcal{I}}(t)\ T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t'\, H_{\mathcal{I}}(t')\right]\right\}.\end{aligned} \tag{3.5.24}$$

In this derivation, we have used the fact that operators inside a time-ordering operator can be written in any order, and that the operator that acts last can be pulled out to the left of the time-ordering.

In the interaction picture, we can express the time evolution of a field through an evolution operator $U_{\mathcal{I}}(t,\, t_0)$,

$$\phi(t,\mathbf{x}) = U_{\mathcal{I}}^{\dagger}(t,t_0)\;\phi_0(t,\mathbf{x})\;U_{\mathcal{I}}(t,t_0) \qquad \Leftrightarrow \qquad \phi(x) = U_{\mathcal{I}}^{\dagger}(t,t_0)\;\phi_0(x)\;U_{\mathcal{I}}(t,t_0)\,. \tag{3.5.25}$$

The operator $U_{\mathcal{I}}(t,t_0)$ is uniquely defined by

$$\begin{aligned} U_{\mathcal{I}}(t,\,t_0) &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \cdots \int_{t_0}^{t} \mathrm{d}t_n\, T\big\{H_{\mathcal{I}}(t_1)H_{\mathcal{I}}(t_2)\cdots H_{\mathcal{I}}(t_n)\big\} \\ &= T\left\{ \exp\left[-i \int_{t_0}^{t} \mathrm{d}t''\, H_{\mathcal{I}}(t'')\right]\right\}, \end{aligned} \tag{3.5.26}$$

where the time-ordering of the exponential is defined as a Taylor series with each term time-ordered. We can generalise these results, and verify that for any two times t_1 and t_2,

$$U_{\mathcal{I}}(t_1,t_2) = e^{iH_0(t_1-t_0)}\; e^{-iH(t_1-t_2)}\; e^{iH_0(t_2-t_0)}\,; \tag{3.5.27}$$

that the initial condition with any arbitrary time gives

$$U_{\mathcal{I}}(t',t') = 1\,; \tag{3.5.28}$$

and that for any three times t_1, t_2 and t_3 we have

$$U_{\mathcal{I}}(t_1,t_3) = U_{\mathcal{I}}(t_1,t_2)\;U_{\mathcal{I}}(t_2,t_3)\,. \tag{3.5.29}$$

Exercise 3.12. In this exercise, we make use of the differential equations (3.5.12) and (3.5.17), to which the free evolution operator $U_0(t,t_0)$ and the full evolution operator $U_{\text{full}}(t,t_0)$ respectively obey, and relate the interaction-picture propagator $U_{\mathcal{I}}(t,t_0)$ to the interaction-picture Hamiltonian H_I.

1. Show that the operator $U_{\mathcal{I}}(t,t_0)$ satisfies

$$i\frac{\partial}{\partial t}U_{\mathcal{I}}(t,t_0) = H_{\mathcal{I}}(t)\;U_{\mathcal{I}}(t,t_0) \qquad \text{with} \qquad U_{\mathcal{I}}(t_0,t_0) = 1\,.$$

2. Demonstrate that the solution to this equation is given by

$$U_{\mathcal{I}}(t,\,t_0) = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n\, H_{\mathcal{I}}(t_1)H_{\mathcal{I}}(t_2)\cdots H_{\mathcal{I}}(t_n)\,.$$

3. Show that this solution can be rewritten as

$$U_{\mathcal{I}}(t,\,t_0) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \cdots \int_{t_0}^{t} \mathrm{d}t_n\, T\big\{H_{\mathcal{I}}(t_1)H_{\mathcal{I}}(t_2)\cdots H_{\mathcal{I}}(t_n)\big\}\,,$$

   and conclude that the operator $U_{\mathcal{I}}(t,\,t_0)$ can be written as a time-ordered exponential.

4. Verify that for any arbitrary time t_1 and t_2,

$$U_{\mathcal{I}}(t_1,t_2) = e^{iH_0(t_1-t_0)}\; e^{-iH(t_1-t_2)}\; e^{iH_0(t_2-t_0)}\,,$$

   and that for any three times t_1, t_2 and t_3,

$$U_{\mathcal{I}}(t_1,t_3) = U_{\mathcal{I}}(t_1,t_2)\;U_{\mathcal{I}}(t_2,t_3)\,.$$

The interaction-picture time evolution operator $U_{\mathcal{I}}(t,t_0)$ provides a means to express the time evolution of the field $\phi(x)$ in terms of the free field $\phi_0(x)$. We now show that it further relates the free theory vacuum $|0\rangle$ to the interacting theory vacuum $|\Omega\rangle$. We recall that the first vacuum is annihilated, at infinitely early times, by the annihilation operator included in the definition of the free field $\phi_0(x)$, and that the second vacuum is annihilated, also at infinitely early times, by the annihilation operators included in the field $\phi(x)$. However, a vacuum state defined as an empty state (in which there is no particle) at a given time usually does not stay empty as time evolves when interactions are turned on.

We recall that the evolution of the two vacua from the reference time t_0 to a specific time t is given by

$$U_0(t,t_0)|0\rangle \qquad \text{and} \qquad U_{\text{full}}(t,t_0)|\Omega\rangle\,. \tag{3.5.30}$$

In the limit $t\to-\infty$, the two vacua represent the same physical system (in which there are no particles) so that the corresponding quantum state can only differ by its normalisation (according to the first postulate of quantum mechanics),

$$\lim_{t\to-\infty} U_0(t,t_0)|0\rangle \propto \lim_{t\to-\infty} U_{\text{full}}(t,t_0)|\Omega\rangle\,. \tag{3.5.31}$$

We can therefore write

$$|\Omega\rangle = \mathcal{N}\lim_{t\to-\infty}\left[U^\dagger_{\text{full}}(t,t_0)\,U_0(t,t_0)\right]|0\rangle = \mathcal{N}\lim_{t\to-\infty} U^\dagger_{\mathcal{I}}(t,t_0)\,|0\rangle\,, \tag{3.5.32}$$

where we have used (3.5.20), and where $\mathcal{N}$ is a complex number. Moreover, at infinitely late times we can similarly derive

$$\langle\Omega| = \mathcal{N}'\,\langle 0|\lim_{t\to\infty}\left[U^\dagger_0(t,t_0)U_{\text{full}}(t,t_0)\right] = \mathcal{N}'\,\langle 0|\lim_{t\to\infty} U_{\mathcal{I}}(t,t_0)\,, \tag{3.5.33}$$

where $\mathcal{N}'$ is another complex number. These two results, together with the evolution law (3.5.25) satisfied by the field $\phi(x)$, allow us to rewrite (3.5.10) in terms of the free theory vacuum $|0\rangle$ and the free fields $\phi_0(x)$. Since all fields lie within a time-ordered product, we can assume that $t_1>t_2$ without any loss of generality. This gives

$$\begin{aligned}
&\langle\Omega\,|\,T\{\phi(x_2)\phi(x_1)\}\,|\,\Omega\rangle \\
&\quad= \mathcal{N}\mathcal{N}'\,\langle 0\,|\,U_{\mathcal{I}}(\infty,t_0)\;U^\dagger_{\mathcal{I}}(t_2,t_0)\phi_0(x_2)U_{\mathcal{I}}(t_2,t_0)\;U^\dagger_{\mathcal{I}}(t_1,t_0)\phi_0(x_1)U_{\mathcal{I}}(t_1,t_0)\;U^\dagger_{\mathcal{I}}(-\infty,t_0)\,|\,0\rangle \\
&\quad= \mathcal{N}\mathcal{N}'\,\langle 0\,|\,U_{\mathcal{I}}(\infty,t_2)\;\phi_0(x_2)\;U_{\mathcal{I}}(t_2,t_1)\;\phi_0(x_1)\;U_{\mathcal{I}}(t_1,-\infty)\,|\,0\rangle\,,
\end{aligned} \tag{3.5.34}$$

after introducing the abuse of notation

$$U_{\mathcal{I}}(\pm\infty,t_0) = \lim_{t\to\pm\infty} U_{\mathcal{I}}(t,t_0)\,. \tag{3.5.35}$$

For the second equality, we have made use of the fact that the evolution operator satisfies the following two properties,

$$U^\dagger_{\mathcal{I}}(t,t_0) = U_{\mathcal{I}}(t_0,t) \qquad \text{and} \qquad U_{\mathcal{I}}(t_1,t_3) = U_{\mathcal{I}}(t_1,t_2)\;U_{\mathcal{I}}(t_2,t_3) \quad \forall\; t_1,t_2,t_3\,. \tag{3.5.36}$$

As the definition (3.5.26) of the evolution operator involves a time-ordered product, all operators included in (3.5.34) are time-ordered. Consequently, we can insert again the time-ordering operator and equivalently write (3.5.34) as

$$\langle\Omega\,|\,T\{\phi(x_2)\phi(x_1)\}\,|\,\Omega\rangle = \mathcal{N}\mathcal{N}'\,\Big\langle 0\,\Big|\,T\{U_{\mathcal{I}}(\infty,t_2)\;\phi_0(x_2)\;U_{\mathcal{I}}(t_2,t_1)\;\phi_0(x_1)\;U_{\mathcal{I}}(t_1,-\infty)\}\,\Big|\,0\Big\rangle\,. \tag{3.5.37}$$

The explicit appearance of the time-ordering operator next allows us to re-order the fields and evolution operators which we can combine by virtue of (3.5.36),

$$\langle\Omega\,|\,T\{\phi(x_2)\phi(x_1)\}\,|\,\Omega\rangle = \mathcal{N}\mathcal{N}'\,\Big\langle 0\,\Big|\,T\{\phi_0(x_2)\;\phi_0(x_1)\;U_{\mathcal{I}}(\infty,-\infty)\}\,\Big|\,0\Big\rangle\,. \tag{3.5.38}$$

We are thus left with the determination of the normalisation factor $\mathcal{N}\mathcal{N}'$ in terms of the fields. This can be achieved from the normalisation condition of the vacuum state,

$$\langle\Omega|\Omega\rangle = 1 \qquad \Leftrightarrow \qquad \mathcal{N}\mathcal{N}' = \frac{1}{\langle 0\,|U_{\mathcal{I}}(\infty,-\infty)|\,0\rangle}\,. \tag{3.5.39}$$

The quantity $\langle 0|U_{\mathcal{I}}(\infty,-\infty)|0\rangle$ is called the *vacuum persistence amplitude*, and is used to normalise all matrix elements to be calculated. This yields

$$\langle \Omega \,|\, T\{\phi(x_2)\phi(x_1)\} \,|\, \Omega\rangle = \frac{\langle 0 \,|\, T\{\phi_0(x_2)\ \phi_0(x_1)\ U_{\mathcal{I}}(\infty,-\infty)\} \,|\, 0\rangle}{\langle 0 \,|U_{\mathcal{I}}(\infty,-\infty)|\, 0\rangle}\,. \tag{3.5.40}$$

This formula can be further simplified by working out an expression for the operator $U_{\mathcal{I}}(\infty,-\infty)$. We start from its definition (3.5.26) and that of the interaction picture Hamiltonian $H_{\mathcal{I}}(t)$ given in (3.5.14),

$$U_{\mathcal{I}}(\infty,-\infty) = T\left\{ \exp\left[-i\int_{-\infty}^{\infty} \mathrm{d}t\, H_{\mathcal{I}}(t)\right]\right\} = T\left\{ \exp\left[-i\int_{-\infty}^{\infty} \mathrm{d}t\, \left[U_0^\dagger(t,t_0)\, H_{\text{int}}\, U_0(t,t_0)\right]\right]\right\}. \tag{3.5.41}$$

The Hamiltonian H_{int} is time-independent, since it is expressed in terms of fields evaluated at the reference time t_0. At this time, the free and interacting fields are the same, which shows that the interaction-picture Hamiltonian $H_{\mathcal{I}}(t)$ can be expressed entirely in terms of free fields at all times. By definition, the full Hamiltonian includes kinetic terms, mass terms and interaction terms. The first two sets of terms are also part of the free theory, while the latter set of terms forms the interaction potential. We can then thus rewrite $H_{\text{int}}[\phi(x)]$ in terms of the interacting part of the corresponding Lorentz-invariant Lagrangian density $\mathcal{L}_{\text{int}}$ defined as the difference between the full Lagrangian density and the free Lagrangian comprising kinetic and mass terms,

$$H_{\text{int}} = -\int_{\mathbb{R}^3} \mathrm{d}^3x\, \mathcal{L}_{\text{int}}[\phi_0(t_0,\mathbf{x})]\,. \tag{3.5.42}$$

In this expression, we explicitly specified that the interaction Lagrangian $\mathcal{L}_{\text{int}}[\phi_0(t_0,\mathbf{x})]$ was written in terms of free fields. This therefore yields

$$U_{\mathcal{I}}(\infty,-\infty) = T\left\{ \exp\left[i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\ \mathcal{L}_{\text{int}}[\phi_0(x)]\right]\right\}, \tag{3.5.43}$$

where all time-evolution operators have been applied directly to the fields appearing in the Lagrangian. The result (3.5.40) can thus be rewritten in a manifestly Lorentz-invariant way,

$$\langle \Omega \,|\, T\{\, \phi(x_2)\phi(x_1)\} \,|\, \Omega\rangle = \frac{\langle 0 \,|\, T\{\phi_0(x_2)\ \phi_0(x_1)\ e^{i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\ \mathcal{L}_{\text{int}}[\phi_0(x)]}\} \,|\, 0\rangle}{\langle 0 \,|T\{\, e^{i\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\ \mathcal{L}_{\text{int}}[\phi_0(x)]}\} \,|\, 0\rangle}\,. \tag{3.5.44}$$

The above formula is immediately generalised to correlation functions involving arbitrarily many fields: for any extra field factor in the left-hand side of the equation, we get an extra free field factor on its right-hand side.

Elements of the S-matrix involved in the LSZ formula (3.4.12) involve correlation functions of the form $\langle\Omega|\, T\{\phi(x_n)\dots\phi(x_1)\}\,|\Omega\rangle$. These can be written in a compact and manifestly Lorentz-invariant manner as

$$\left\langle \Omega \,\middle|\, T\{\ \phi(x_n)\dots\phi(x_1)\ \}\,\middle|\, \Omega\right\rangle = \frac{\langle 0 \,|\, T\{\phi_0(x_n)\ \dots\phi_0(x_1)\ e^{i\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}[\phi_0(x)]}\} \,|\, 0\rangle}{\langle 0 \,|T\{\, e^{i\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}[\phi_0(x)]}\} \,|\, 0\rangle}\,, \tag{3.5.45}$$

where the integrals are evaluated over the entire space-time. While this expression is exact, it is also suitable for perturbative calculations. In this case, we perform a Taylor series expansion of the exponentials and retain as many terms as desired.

As a consequence of (3.5.45), any practical calculation amounts to the evaluation of vacuum expectation values of products of free fields like $\langle 0|\, T\{\phi(x_n)\dots\phi(x_1)\}\,|0\rangle$. It is possible to do it in a brute-force way by means of the ladder operator algebra (as done for the Feynman propagator earlier in this section), but there exists a much more practical and less cumbersome manner relying on Feynman diagrams and *Wick's theorem*.

3.6 Feynman rules and Feynman diagrams

We begin this section by considering again the Feynman propagator (3.5.2), that consists of the simplest correlation function that could be calculated,

$$G_F(x_2-x_1)=\langle 0\,|\,T\{\phi_0(x_2)\phi_0(x_1)\}\,|\,0\rangle\,. \tag{3.6.1}$$

We will, however, make use this time of a technique different from the brute-force one of section 3.5 to calculate it, as the latter does not generalise well to higher-point correlators. In order to do so, we rewrite the free field (3.5.1) as a sum of an annihilation part and a creation part,

$$\phi_0(x)=\phi_0^{\rm cr}(x)+\phi_0^{\rm an}(x)\,, \tag{3.6.2}$$

with

$$\phi_0^{\rm cr}(x)=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,a_{\mathbf{p}}^\dagger\,e^{ip\cdot x}\qquad\text{and}\qquad\phi_0(x)^{\rm an}=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,a_{\mathbf{p}}\,e^{-ip\cdot x}\,. \tag{3.6.3}$$

One of the key properties of the objects $\phi_0^{\rm cr}(x)$ and $\phi_0^{\rm an}(x)$ is that they satisfy

$$\langle 0\,|\,\phi_0^{\rm cr}(x)=0\qquad\text{and}\qquad\phi_0^{\rm an}(x)\,|\,0\rangle=0\,. \tag{3.6.4}$$

With this decomposition, the time-ordered product of two fields reads, for $t_2>t_1$,

$$\begin{aligned}T\{\phi_0(x_2)\phi_0(x_1)\}&\underset{t_2>t_1}{=}\phi_0^{\rm cr}(x_2)\phi_0^{\rm cr}(x_1)+\phi_0^{\rm an}(x_2)\phi_0^{\rm cr}(x_1)+\phi_0^{\rm cr}(x_2)\phi_0^{\rm an}(x_1)+\phi_0^{\rm an}(x_2)\phi_0^{\rm an}(x_1)\\&=\phi_0^{\rm cr}(x_2)\phi_0^{\rm cr}(x_1)+\phi_0^{\rm cr}(x_1)\phi_0^{\rm an}(x_2)+\phi_0^{\rm cr}(x_2)\phi_0^{\rm an}(x_1)+\phi_0^{\rm an}(x_2)\phi_0^{\rm an}(x_1)\\&\quad+\big[\phi_0^{\rm an}(x_2),\phi_0^{\rm cr}(x_1)\big]\,.\end{aligned} \tag{3.6.5}$$

To derive the second equality, we have re-ordered all terms of the first equality such that creation operators appear on the left of annihilation operators, which yielded an extra term with a commutator. We recall that the relative ordering of creation operators among themselves is irrelevant as they commute, as for the annihilation operators among themselves. Such an ordering of annihilation and creation operators is said to be a *normal ordering*, and the resulting operator has a vanishing vacuum expectation value according to (3.6.4). Moreover, the case of $t_1<t_2$ leads to a similar result as in (3.6.5),

$$\begin{aligned}T\{\phi_0(x_2)\phi_0(x_1)\}&\underset{t_1>t_2}{=}\phi_0^{\rm cr}(x_2)\phi_0^{\rm cr}(x_1)+\phi_0^{\rm cr}(x_2)\phi_0^{\rm an}(x_1)+\phi_0^{\rm cr}(x_1)\phi_0^{\rm an}(x_2)+\phi_0^{\rm an}(x_2)\phi_0^{\rm an}(x_1)\\&\quad+\big[\phi_0^{\rm an}(x_1),\phi_0^{\rm cr}(x_2)\big]\,.\end{aligned} \tag{3.6.6}$$

We can therefore rewrite the operator $T\{\phi_0(x_2)\phi_0(x_1)\}$ in a compact way after introducing the *normal ordering operator* $\mathcal{N}(\phi_0(x_2)\phi_0(x_1))$, that is also sometimes written as $:\phi_0(x_2)\phi_0(x_1):$. This operator applied on a product of ladder operators places all creation and annihilation operators appearing in the product such that the annihilation operators lie on the right of creation operators regardless of the commutation relations. In other words, products of annihilation and creation operators are rewritten 'manually' by permuting the operators and placing all creation operators on the right of all annihilation operators. The permutation being made by ignoring the non-trivial structure of the commutation relations (2.5.12). This guarantees that the corresponding vacuum expectation value vanishes by virtue of (3.6.4),

$$\Big\langle 0\,\Big|\,\mathcal{N}\big(\text{any product of } a \text{ and } a^\dagger\big)\,\Big|\,0\Big\rangle=0\,. \tag{3.6.7}$$

Combining (3.6.5) and (3.6.6), we get

$$\begin{aligned}T\{\phi_0(x_2)\phi_0(x_1)\}&=\mathcal{N}\Big(\phi_0(x_2)\phi_0(x_1)\Big)+\big[\phi_0^{\rm an}(x_1),\phi_0^{\rm cr}(x_2)\big]\theta(t_1-t_2)+\big[\phi_0^{\rm an}(x_2),\phi_0^{\rm cr}(x_1)\big]\theta(t_2-t_1)\\&=\mathcal{N}\Big(\phi_0(x_2)\phi_0(x_1)+\overbracket{\phi_0(x_2)\phi_0(x_1)}\Big)\\&=\mathcal{N}\Big(\phi_0(x_2)\phi_0(x_1)\;+\;G_F(x_2-x_1)\Big)\,.\end{aligned} \tag{3.6.8}$$

The last two terms that appear on the right-hand side of the first equality form what is called a *contraction*,

$$\overbracket{\phi_0(x_2)\phi_0(x_1)}\equiv\big[\phi_0^{\rm an}(x_1),\phi_0^{\rm cr}(x_2)\big]\theta(t_1-t_2)+\big[\phi_0^{\rm an}(x_2),\phi_0^{\rm cr}(x_1)\big]\theta(t_2-t_1)\,, \tag{3.6.9}$$

that is exactly given by the Feynman propagator $G_F(x_2-x_1)$,

$$\overbracket{\phi_0(x_2)\phi_0(x_1)}=G_F(x_2-x_1)\,. \tag{3.6.10}$$

Wick's theorem generalises the result in (3.6.8) to the case of a product of an arbitrary number of fields.

Wick's theorem relates time-ordered products of fields to normal-ordered products of fields and contractions. A contraction consists of the connection of two fields from the product that we replace by a Feynman propagator. We have:

$$T\Big\{\ \phi(x_n)\dots\phi(x_1)\ \Big\} = \mathcal{N}\Big(\phi_0(x_n)\dots\phi_0(x_1)\ +\ \text{all possible contractions}\Big)\,. \tag{3.6.11}$$

Here, 'all possible contractions' means that we must include terms involving any of the fields of the product and including one contraction, two contractions, *etc.*

Exercise 3.13. Demonstrate Wick's theorem by induction.

For instance, applying Wick's theorem to a product of $n = 4$ fields leads to

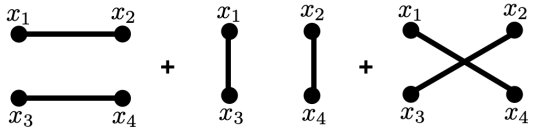
$$\begin{aligned} T\{\phi_4\phi_3\phi_2\phi_1\} = \mathcal{N}\Big(&\phi_4\phi_3\phi_2\phi_1 + \overbrace{\phi_4\phi_3}\phi_2\phi_1 + \overbrace{\phi_4\phi_3\phi_2}\phi_1 + \overbrace{\phi_4\phi_3\phi_2\phi_1} + \phi_4\overbrace{\phi_3\phi_2}\phi_1 + \phi_4\overbrace{\phi_3\phi_2\phi_1} \\ &+ \phi_4\phi_3\overbrace{\phi_2\phi_1} + \overbrace{\phi_4}\phi_3\overbrace{\phi_2}\phi_1 + \overbrace{\phi_4\phi_3}\overbrace{\phi_2\phi_1} + \overbrace{\phi_4\phi_3\phi_2}\overbrace{\phi_1}\Big)\,, \end{aligned} \tag{3.6.12}$$

in which we have introduced the shortcut notation $\phi_i \equiv \phi(x_i)$. Only terms in which all fields are contracted contribute the associated vacuum expectation value, by virtue of the properties of the normal-ordering operator. We therefore get, after using (3.6.10) and introducing a shorthand notation of the Feynman propagator $G_{ij} \equiv G_F(x_i - x_j)$,

$$\big\langle 0\,\big|\,T\{\phi_4\phi_3\phi_2\phi_1\}\,\big|\,0\big\rangle = G_{43}G_{21} + G_{42}G_{31} + G_{41}G_{32}\,. \tag{3.6.13}$$

Each of these terms can be drawn as a diagram in which the space-time points x_1, x_2, x_3 and x_4 at which the correlation functions are evaluated are represented by dots, and in which each Feynman propagator G_{ij} is represented by a line joining the dots associated with x_i and x_j. Applying these rules to the above expression, we get

$$\big\langle 0\,\big|\,T\{\phi_4\phi_3\phi_2\phi_1\}\,\big|\,0\big\rangle = $$



. (3.6.14)

The diagrams do not represent any measurable quantity, however, they suggest a straightforward interpretation: Two particles are created at two space-time points, and then propagate to two other phase-space points at which they are annihilated. This phenomenon can happen in three different ways when four phase-space points are considered so that the full amplitude is given by the sum over these three ways. These diagrams are called *Feynman diagrams*.

Feynman diagrams become more interesting once several fields are evaluated at a specific space-time point. To this aim, we consider a three-point interaction modelled by the interaction Lagrangian

$$\mathcal{L}_{\rm int}[\phi] = \frac{1}{3!}\lambda_3\phi_0^3\,, \tag{3.6.15}$$

where the parameter λ_3 represents the strength of the interaction and is sufficiently small to justify the usage of perturbation theory for the calculation of any correlation function. We moreover included in (3.6.15) a normalisation factor of $1/3!$ to account for the fact that the interaction Lagrangian term involves the same field three times. This factor is purely combinatorial, and the exact reason for its inclusion will become clearer below.

We now go back to the example of a two-point correlation function, and we focus first on the numerator of (3.5.45) that is relevant for the calculation of $\langle\Omega|T\{\phi_0(x_2)\phi_0(x_1)|\Omega\rangle$. This requires us to calculate, at

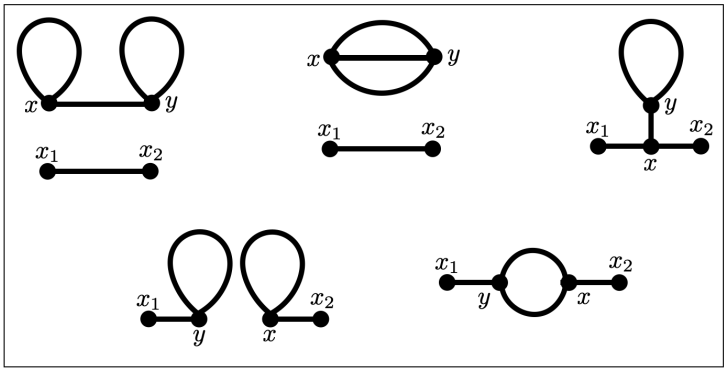


Figure 3.5: Feynman diagrams associated with the $\mathcal{O}(\lambda_3^2)$ contributions to the two-point correlation function $\langle 0|T\{\phi_0(x_2)\phi_0(x_1)e^{-i\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}[\phi_0(x)]}\}|0\rangle$.

order λ_3^2, the quantity

$$\begin{aligned}\left\langle 0 \,\middle|\, T\{\phi_0(x_2)\phi_0(x_1)\; e^{-i\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{int}}[\phi_0(x)]}\;\}\,\middle|\, 0\right\rangle &= \langle 0|T\{\phi_2\phi_1\}|0\rangle + \frac{i\lambda_3}{3!}\int \mathrm{d}^4x\,\langle 0|T\{\phi_2\phi_1\phi_x^3\}|0\rangle \\ &- \frac{\lambda_3^2}{2(3!)^2}\int \mathrm{d}^4x\int \mathrm{d}^4y\,\left\langle 0 \,\middle|\, T\{\phi_2\phi_1\phi_x^3\phi_y^3\}\,\middle|\, 0\right\rangle + \mathcal{O}(\lambda_3^3)\,,\end{aligned} \tag{3.6.16}$$

in which we again make use of the notation $\phi_i = \phi_0(x_i)$. The first term on the right-hand side of this equation is the one that we have already calculated in section 3.5 and in the beginning of this section,

$$\left\langle 0 \,\middle|\, T\{\phi_2\phi_1\}\,\middle|\, 0\right\rangle = G_{21}\,, \tag{3.6.17}$$

where we again have $G_{ij} = G_F(x_i - x_j)$. This term is represented diagrammatically by

$$\left\langle 0 \,\middle|\, T\{\phi_2\phi_1\}\,\middle|\, 0\right\rangle = \quad . \tag{3.6.18}$$

The next term in (3.6.16) contains an odd number of fields and cannot therefore be fully contracted. Consequently, by virtue of Wick's theorem and the properties of the normal-ordering operator when vacuum expectation values are computed, it vanishes. Finally, the last term involves eight occurrences of the field operator, and it is therefore associated with multiple possible contractions. Only a few of them are however different, and we additionally have solely to focus on contributions that are fully contracted. Any other, not fully contracted, term will lead to a vanishing contribution to the vacuum expectation value as a consequence of the properties of the normal-ordering operator. This leads to eight different contributions, that are given by

$$\begin{aligned}\langle 0|T\{\phi_2\phi_1\phi_x^3\phi_y^3\}|0\rangle &= 9\,G_{21}G_{xx}G_{yy}G_{xy} + 6\,G_{21}G_{xy}^3 + 18\,G_{2x}G_{1x}G_{yy}G_{xy} + 9\,G_{2x}G_{1y}G_{yy}G_{xx} \\ &+ 18\,G_{2x}G_{1y}G_{xy}^2 + 18\,G_{2y}G_{1y}G_{xx}G_{xy} + 9\,G_{2y}G_{1x}G_{yy}G_{xx} + 18\,G_{2y}G_{1x}G_{xy}^2\,.\end{aligned} \tag{3.6.19}$$

The symmetry properties arising from the fact that x and y integration is carried out over the entire space-time ensures that the last three terms are respectively identical to the third, fourth and fifth terms in (3.6.19). We can therefore simplify this relation as

$$\begin{aligned}-\frac{\lambda_3^2}{2(3!)^2}\int \mathrm{d}^4x\int \mathrm{d}^4y\,\left\langle 0 \,\middle|\, T\{\phi_2\phi_1\phi_x^3\phi_y^3\}\,\middle|\, 0\right\rangle &= -\lambda_3^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\Big(\frac{1}{8}\,G_{21}G_{xx}G_{yy}G_{xy} \\ &+ \frac{1}{12}\,G_{21}G_{xy}^3 + \frac{1}{2}\,G_{2x}G_{1x}G_{yy}G_{xy} + \frac{1}{4}\,G_{2x}G_{1y}G_{yy}G_{xx} + \frac{1}{2}\,G_{2x}G_{1y}G_{xy}^2\Big)\,.\end{aligned} \tag{3.6.20}$$

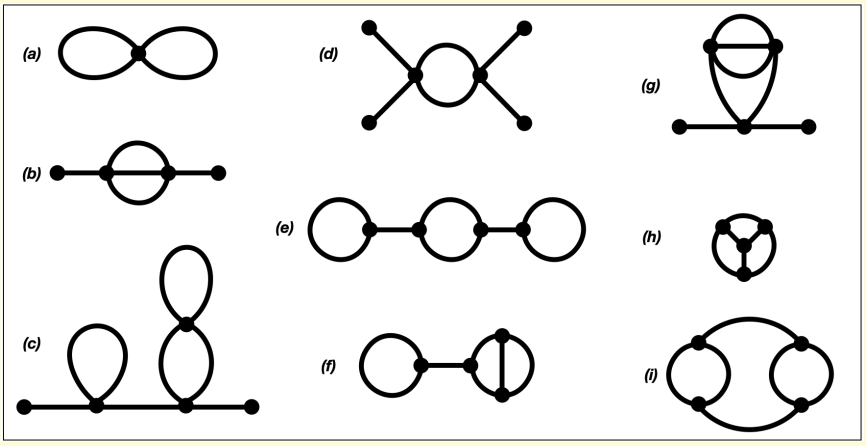
Diagrammatically, we thus have a total of five independent contributions to the two-point correlation function, that all come with different combinatorial factors. The corresponding diagrams are given in figure 3.5. From these diagrams, we can straightforwardly infer the way in which the perturbative expansion works for higher-order terms or more general interactions. This way is known as *position-space Feynman rules*.

In the general case, all contributions to an n-point correlation function can be computed diagrammatically and written as space-time integrals over a sum of products of Feynman propagators. The corresponding analytic expression is deduced as follows.

1. We begin with external points x_i associated to each position at which the fields in the correlation function are evaluated.
2. We next draw a line starting at each point.
3. Each line can then either contract to an existing line (yielding a Feynman propagator) or split due to an interaction. Such a splitting gives rise to a new *vertex* with a strength extracted from the interaction Lagrangian ($i\lambda_3$ in our case), as well as an integral over the space-time position of this vertex (we have two vertices in the example above, that we labeled by the space-time coordinates x and y; this yielded two space-time integrals over x and y).
4. The contribution of each diagram must then be divided by its geometrical symmetry factor, which corresponds to the number of ways in which a Feynman diagram can be deformed so that it looks the same as when the external points are held fixed [26]. This includes factors of 2 for each line that starts and ends at the same vertex (the diagram being symmetric under the exchange of the two ends of such a line), for each set of two propagators connecting two specific vertices (the diagram being symmetric under the exchange of these propagators), and for each set of equivalent vertices (the diagram being symmetric under the exchange of these vertices).
5. The full amplitude at a given order in perturbation theory is provided by the sum of all diagrams.

Following this method, the factorial factors appearing both in the interaction Lagrangian and in the series expansion of the exponential can be ignored, as they are cancelled by the symmetry factors associated with the number of permutations of lines and vertices leaving a given diagram unchanged. More precisely, at order m we have m internal vertices in the diagram, that we can permute in $m!$ possible ways. This $m!$ factor cancels the $m!$ factor originating from the Taylor series expansion of the exponential. Furthermore, we normalise each Lagrangian term such that it includes a $1/j!$ factor for each set of j identical fields. This factor is cancelled by the $j!$ ways to permute the identical lines arriving at a given vertex.

Exercise 3.14. Determine the symmetry factors of the following Feynman diagrams.



Exercise 3.15. Demonstrate that for an interaction Lagrangian

$$\mathcal{L}_{\text{int}}[\phi_0] = \frac{1}{3!}\lambda_3\phi_0^3\,,$$

there are only five different position-space Feynman diagrams that contribute to the two-point correlation functions at order λ_3^2. Find these diagrams and estimate the strength of each of them (with the correct symmetry factor included).

In order to calculate correlation functions $\langle\Omega|\,T\{\phi(x_n)\dots\phi(x_1)\}\,|\Omega\rangle$ in the interacting theory, we also need to evaluate the denominator of (3.5.45). This can be achieved in a similar manner as to what we

did above. At order λ_3^2 we hence get,

$$\left[\left\langle 0\left|T\left\{\, e^{i\int \mathrm{d}^4x\ \mathcal{L}_{\mathrm{int}}[\phi_x]}\,\right\}\right|0\right\rangle\right]^{-1} = 1 + \frac{\lambda_3^2}{2(3!)^2}\int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{1}{8}\, G_{xx}G_{yy}G_{xy} + \frac{1}{12}\, G_{xy}^3\right). \tag{3.6.21}$$

We can now go back to the example of the two-point function and plug everything in together. We get, from (3.6.17), (3.6.20) and (3.6.21),

$$\begin{aligned}\left\langle \Omega\left|T\left\{\ \phi(x_n)\dots\phi(x_1)\ \right\}\right|\Omega\right\rangle &= G_{21} \\ &- \lambda_3^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{1}{2}\, G_{2x}G_{1x}G_{yy}G_{xy} + \frac{1}{4}\, G_{2x}G_{1y}G_{yy}G_{xx} + \frac{1}{2}\, G_{2x}G_{1y}G_{xy}^2\right).\end{aligned} \tag{3.6.22}$$

The effect of the denominator of (3.5.45) is to cancel the contributions of the first two diagrams in figure 3.5, that correspond to diagrams involving at least one sub-graph that is not connected to any of the external points (x_1 and x_2 here). Such graphs are called *vacuum bubbles*, and their contributions to correlation functions always cancel. An integral that involves a sub-graph not connected to any of the external points always factors out, and the cancellation then arises from the fact that this factor also appears in the denominator of (3.5.45).

Among the three remaining Feynman diagrams of figure 3.5 (*i.e.* the last three diagrams in the figure), the second one is special in the sense that it comprises two *disconnected sub-graphs*. These sub-graphs do not consist of bubble diagrams as each of them is connected to at least one external point. These diagrams cannot therefore be removed from the calculation of an element of the S-matrix, and they physically contribute to the total amplitude like any other fully *connected diagram* (such as the third and last diagrams in figure 3.5). Such disconnected diagrams can, however, be calculated independently by virtue of the *cluster decomposition principle* [27, 28]. Their contribution to the matrix element squared can be seen as emerging from the product of contributions of two independent scattering processes involving a smaller number of particles. Naively, we could imagine that they should interfere with the connected diagrams so that they cannot be considered independently. The cluster decomposition principle states that this is not the case, and that scattering processes well-separated in space cannot interfere. This principle is often considered to be an axiom of quantum field theory, and it tells us that elements of the S-matrix (with bubbles removed) factorises into products of sums of connected diagrams. Consequently, practical calculations only require us to calculated connected diagrams, possibly with a smaller number of external points.

Correlation functions $\langle\Omega|\, T\{\phi(x_n)\dots\phi(x_1)\}\,|\Omega\rangle$ can be determined from position-space Feynman diagrams and Feynman rules, through the calculation of vacuum expectation values of time-ordered products of fields of the form

$$\left\langle 0\left|T\left\{\phi_0(x_m)\ \dots\phi_0(x_1)\ \right\}\right|0\right\rangle_{\text{connected}}. \tag{3.6.23}$$

The subscript 'connected' indicates that disconnected diagrams (*i.e.* diagrams that contains a sub-graph that is not connected to all external points $x_1,\dots,x_m$) are ignored. The above quantity can be calculated order by order in perturbation theory through the method of *Feynman diagrams*:

1. We draw one dot for each of the m external points at which the fields in the correlation function are evaluated.
2. We draw a line starting at each dot.
3. Each line can then either contract to an existing line, or it can split at a new vertex due to an interaction of the Lagrangian. The number of lines originating from a splitting corresponds to the number of fields in the related Lagrangian term, and the maximum number of vertices allowed in a diagram is fixed by the perturbative order to which the calculation is achieved. We assign a new space-time position to each introduced vertex.
4. The full amplitude to a given order in perturbation theory is the sum of all diagrams. Only diagrams that are fully connected must be considered, although disconnected diagrams in which each sub-graph is connected to at least one external point have to be considered independently.

Once all diagrams have been drawn, we need to convert them into an analytic expression for the correlation function.

Feynman rules relate each diagram to an analytic expression providing its contribution to $\langle 0|T\{\phi_0(x_m)\dots\phi_0(x_1)\}|0\rangle_{\text{connected}}$.

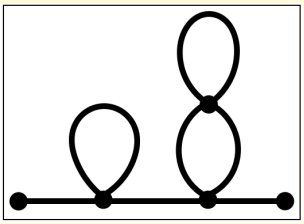
1. Each internal line between two space-time points x and y yields a Feynman propagator $G_F(x-y)$.
2. If the Lagrangian terms associated with a given vertex has a $\lambda/j!$ prefactor, with j being the number of identical fields incoming to that vertex, the analytic expression for $\langle 0|T\{\phi_0(x_m)\dots\phi_0(x_1)\}|0\rangle_{\text{connected}}$ includes a factor of $i\lambda$. Moreover, we also need to add an x-integral over the entire space-time.
3. Each external point is replaced by 1.
4. The contribution of each diagram is divided by its geometrical symmetry factor.

Position-space Feynman rules have a simple quantum mechanical interpretation that relies on the principle of superposition when we consider that each vertex in a graph is associated with the amplitude $i\lambda$ to absorb and emit particles. The additional integral indicates that this can happen at any space-time point so that we need to sum over all possibilities. Furthermore, when a process can occur through different manners (*i.e.* it is related to several Feynman diagrams), we sum over all individual amplitudes.

Exercise 3.16. Consider a theory describing the dynamics of a scalar field ϕ whose interaction Lagrangian is given by

$$\mathcal{L}_{\text{int}}[\phi] = -\frac{\lambda_4}{4!}\phi^4\,.$$

Determine the contribution of the cactus diagram (see the figure) to the two-point correlation function.



The position-space Feynman rules discussed so far provide a recipe to calculate the vacuum expectation value of time-ordered products of fields in QFT. These can be used in a second step to calculate elements of the S-matrix according to the LSZ formula (3.4.12), and therefore decay rates and cross sections. The latter are the goals of most QFT calculations as the associated predictions are testable experimentally, and therefore allow for a test of particle physics models against data. The corresponding formulas (3.2.76) and (3.3.2) involve matrix elements $\langle f|\mathcal{M}|i\rangle$ in which the kinematics (and more precisely four-momentum conservation) are factorised out of the S-matrix and are included in the phase space factor. This can also be done from the expressions that we have obtained so far and the LSZ formula, that turn out to be further simplified thanks to this procedure. This relies on the notion of *momentum-space Feynman rules* that we now define on the basis of a specific example that will next be generalised.

We continue with the study of the two-point function evaluated at $\mathcal{O}(\lambda_3^2)$ that we initiated above, and we focus for now on the last diagram of figure 3.5,

$\mathcal{T}_1 =$ 

. (3.6.24)

The corresponding analytic expression can be either extracted from (3.6.20), or directly obtained by employing position-space Feynman rules. It reads

$$\mathcal{T}_1 = -\lambda_3^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\, \frac{1}{2}\, G_{2x}\, G_{1y}\, G_{xy}^2\,. \tag{3.6.25}$$

In order to relate this expression to the matrix element $\langle f|\mathcal{M}|i\rangle$ appearing in cross sections and decay rates, we need to factorise from this quantity four-momentum conservation, *i.e.* a delta function of the

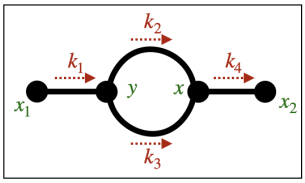


Figure 3.6: Feynman diagram contributing to the two-point function $\langle\Omega|\, T\{\phi(x_2)\phi(x_1)\}\,|\Omega\rangle$ at next-to-leading order. The space-time positions of each vertex is indicated in green, and the four-momenta flowing along each line are shown, together with the direction of the flows, in red. We note that the choice of flow is arbitrary. Other possibilities are allowed, and this will impact the signs appearing in the exponential factors of (3.6.26).

four-momenta. This is achieved by replacing the four Feynman propagators in (3.6.25) with their integral representation (3.5.7),

$$\begin{aligned}\mathcal{T}_1 = &-\frac{\lambda_3^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\int \frac{\mathrm{d}^4k_3}{(2\pi)^4}\int \frac{\mathrm{d}^4k_4}{(2\pi)^4}\; e^{ik_1\cdot(x_1-y)}e^{ik_2\cdot(x-x_2)}\\ &\times e^{ik_3\cdot(y-x)}e^{ik_4\cdot(y-x)}\;\frac{i}{k_1^2-m^2+i\varepsilon}\;\frac{i}{k_2^2-m^2+i\varepsilon}\;\frac{i}{k_3^2-m^2+i\varepsilon}\;\frac{i}{k_4^2-m^2+i\varepsilon}\,.\end{aligned} \tag{3.6.26}$$

We now have four d^4k integrals over the four four-momenta k_1, k_2, k_3 and k_4, that are related to the four Feynman propagators included in (3.6.25). Moreover, we have arbitrarily fixed the flow of the four-momenta such that all four-momenta are flowing from the left to the right of the diagram, which dictates the different relative signs in the exponential factors. More precisely, the four-momentum k_1 is outgoing from x_1 and incoming to y, the four-momentum k_2 is outgoing from x and incoming to x_2, and the two four-momenta k_3 and k_4 are both outgoing from y and incoming to x. This flow is illustrated in figure 3.6.

The integrals over x and y (*i.e.* the space-time coordinates of the two vertices) yield two four-dimensional delta distributions enforcing momentum conservation at each of the vertices x and y,

$$\begin{aligned}\mathcal{T}_1 = &-\frac{\lambda_3^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\int \frac{\mathrm{d}^4k_3}{(2\pi)^4}\int \frac{\mathrm{d}^4k_4}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}\big(k_2-k_3-k_4\big)\,(2\pi)^4\delta^{(4)}\big(k_3+k_4-k_1\big)\\ &\times e^{ik_1\cdot x_1}\;e^{-ik_2\cdot x_2}\;\frac{i}{k_1^2-m^2+i\varepsilon}\;\frac{i}{k_2^2-m^2+i\varepsilon}\;\frac{i}{k_3^2-m^2+i\varepsilon}\;\frac{i}{k_4^2-m^2+i\varepsilon}\,.\end{aligned} \tag{3.6.27}$$

This last result shows that four-momentum is conserved at each of the vertices x and y, which respectively reads $\delta^{(4)}(k_2-k_3-k_4)$ and $\delta^{(4)}(k_3+k_4-k_1)$. We note that the relative signs of the different four-momenta agree with the reference flow chosen in figure 3.6. In order to get a single delta function, we can equivalently integrate over one of the internal loop-momentum k_3 or k_4. Relabelling the other four-momentum as q, we get

$$\begin{aligned}\mathcal{T}_1 = &-\frac{\lambda_3^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\;(2\pi)^4\delta^{(4)}\big(k_2-k_1\big)e^{ik_1\cdot x_1}\;e^{-ik_2\cdot x_2}\\ &\times\frac{i}{k_1^2-m^2+i\varepsilon}\;\frac{i}{k_2^2-m^2+i\varepsilon}\;\frac{i}{(k_1-q)^2-m^2+i\varepsilon}\;\frac{i}{q^2-m^2+i\varepsilon}\,.\end{aligned} \tag{3.6.28}$$

The remaining delta function is precisely the four-momentum conservation factor appearing in the elements of the S-matrix (3.2.66). In the case of a two-point correlation function, the LSZ formula reads

$$\langle f|\,S\,|i\rangle = \Big[i\int \mathrm{d}^4x_2\, e^{ip_f\cdot x_2}\big(\Box_2+m^2\big)\Big]\Big[i\int \mathrm{d}^4x_1\, e^{-ip_i\cdot x_1}\big(\Box_1+m^2\big)\Big]\Big\langle\Omega\Big|T\Big\{\phi(x_2)\phi(x_1)\Big\}\Big|\Omega\Big\rangle\,, \tag{3.6.29}$$

where p_i and p_f stand for the initial-state and final-state four-momenta. The contribution $\langle f|\,S\,|i\rangle_1$ of the diagram $\mathcal{T}_1$ to the matrix element can be obtained by inserting (3.6.28) in the above relation,

$$\begin{aligned}\langle f|\,S\,|i\rangle_1 &= \Big[i\int \mathrm{d}^4x_2\, e^{ip_f\cdot x_2}\big(\Box_2+m^2\big)\Big]\Big[i\int \mathrm{d}^4x_1\, e^{-ip_i\cdot x_1}\big(\Box_1+m^2\big)\Big]\;\mathcal{T}_1\\ &= -\frac{\lambda_3^2}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\;(2\pi)^4\delta^{(4)}\big(p_f-p_i\big)\frac{i}{(p_i-q)^2-m^2+i\varepsilon}\;\frac{i}{q^2-m^2+i\varepsilon}\,.\end{aligned} \tag{3.6.30}$$

The second equality has been derived by first performing the two integrals over x_1 and x_2, which yielded two delta functions $\delta^{(4)}(p_i - k_1)\ \delta^{(4)}(p_f - k_2)$, and then by integrating over k_1 and k_2. This highlights the fact that the LSZ formula enforces the initial-state and final-state particles to be on-shell one-particle states, and that any propagator related to an external leg gets cancelled.

We can extract a formula for the matrix element factor $\langle f|\mathcal{M}|i\rangle$ entering differential cross sections (3.2.76) and decay rates (3.3.2). We recall the definition (3.2.66),

$$\langle f|\, S\,|i\rangle = i\ (2\pi)^4\, \delta^{(4)}\big(p_i - p_f\big)\, \langle f|\, \mathcal{M}\,|i\rangle\,, \tag{3.6.31}$$

which yields, together with (3.6.30),

$$i\mathcal{M}_1 \equiv i\langle f|\, \mathcal{M}\,|i\rangle_1 = -\frac{\lambda_3^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\ \frac{i}{(p_i - q)^2 - m^2 + i\varepsilon}\ \frac{i}{q^2 - m^2 + i\varepsilon}\,. \tag{3.6.32}$$

Here, the notation $\langle f|\, \mathcal{M}\,|i\rangle_1$ indicates that the matrix element is restricted to the contribution of the diagram $\mathcal{T}_1$. We can also notice that this integral is divergent. This feature is properly accounted for in the framework of higher-order calculations in QFT, which goes beyond the scope of this chapter. Instead we now generalise the above derivation to any scattering amplitude, and we provide the so-called *momentum-space Feynman rules*, allowing for the calculation of matrix elements $i\mathcal{M}$ from Feynman diagrams.

We consider processes whose initial state comprises two particles of four-momenta p_a and p_b, and whose final state comprises n particles of four-momenta p_i (with $i = 1, \ldots, n$). The associated element of the S matrix is given by

$$\langle f|\, S\,|i\rangle = i(2\pi)^4\, \delta^{(4)}\left(p_a + p_b - \sum_{i=1}^{n} p_i\right)\, \langle f|\, \mathcal{M}\,|i\rangle \equiv (2\pi)^4\, \delta^{(4)}\left(p_a + p_b - \sum_{i=1}^{n} p_i\right)\, i\mathcal{M}\,. \tag{3.6.33}$$

The matrix element $i\mathcal{M}$ is given by the sum of the contributions of each Feynman diagram associated with the process of interest $i \to f$. The analytic expression to be included for any given diagram is obtained as follows, from the so-called momentum-space Feynman rules:

1. For each internal line, we include the Fourier transform (3.5.8) of the Feynman propagator $\frac{i}{p^2 - m^2 + i\varepsilon}$, where p stands for the four-momentum flowing through the internal line.
2. Lines connected to external points are replaced by 1 (the associated Feynman propagator being cancelled in the LSZ reduction formula).
3. The contributions of each vertex are obtained from the prefactors appearing in the different terms of the interaction Lagrangian. In the example considered, this yielded factors of $i\lambda_3$.
4. The four-momentum flow along all lines of the diagrams is such that momentum conservation is enforced at each vertex.
5. Integration over all undetermined four-momenta must be performed (like q in (3.6.32)).
6. Each contribution must be divided by the appropriate symmetry factor (1/2 in our example).

So far we have considered two types of interaction Lagrangians. The first of these Lagrangians involved a trilinear interaction of the scalar field ϕ, and it was used in the calculation of higher-order corrections to the Feynman propagator. It was given by (3.6.15),

$$\mathcal{L}_{\text{int}}[\phi] = \frac{\lambda_3}{3!}\phi^3\,. \tag{3.6.34}$$

The second Lagrangian that we considered was used for the calculation of the cactus diagram in exercise 3.16. It involved a quartic interaction of the scalar field ϕ,

$$\mathcal{L}_{\text{int}}[\phi] = -\frac{\lambda_4}{4!}\phi^4\,. \tag{3.6.35}$$

In both cases, the Feynman rules associated with the interaction vertices are easy to extract. They read


$\equiv i\lambda_3\,,$

$\equiv -i\lambda_4\,.$ (3.6.36)

These rules consist of the prefactor of the associated Lagrangian terms, times i, and with the combinatorial factor included in the Lagrangian ignored. The latter is always compensated by the symmetry stemming from the permutation of identical lines in these vertices. On the other hand, not including the 1/3! and 1/4! factors in the Lagrangian would have implied including 3! and 4! factors in the respective Feynman rules. This can be generalised when the Lagrangian depends on several fields. In this case, an appropriate combinatorial factor must be included for each set of identical fields.

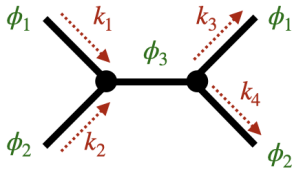
It may happen that some Lagrangian terms include derivative operators, like in

$$\mathcal{L}_{\text{int}}[\phi] = \lambda\ \phi_1\ (\partial^\mu \phi_2)\ (\partial_\mu \phi_3) \equiv \lambda\ \phi_1\ \partial^\mu \phi_2\ \partial_\mu \phi_3\,, \tag{3.6.37}$$

where we consider three different scalar fields ϕ_1, ϕ_2 and ϕ_3 for simplicity (and clarity). In this expression (and in the rest of these notes), the derivative operator only acts on the field directly placed to its right (except in cases where parentheses are explicitly indicating something different). In order to derive the Feynman rule associated to this vertex, we need to resolve the action of the two derivative operators. This can be deduced from the expression of the free field (3.5.1),

$$\phi(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}\ \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\ \left(a_{\mathbf{p}}\ e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}\ e^{ip\cdot x}\right). \tag{3.6.38}$$

This shows that if a particle of four-momentum p^μ is created at a vertex, then the Feynman rule should include a factor of ip^μ, whereas if the particle is annihilated at that vertex, then the Feynman rule should include a factor of $-ip^\mu$. This is sufficient to derive the amplitude associated with any diagram involving the three fields ϕ_1, ϕ_2 and ϕ_3. The only extra rule to consider when drawing the diagrams is that a line related to a given field can only be connected to a line related to the same field. For instance, we have



$$\begin{aligned} &\equiv \Big[(i\lambda)(-ik_{2\mu})i(k_1^\mu + k_2^\mu)\Big]\ \frac{i}{s - m_3^2 + i\varepsilon}\Big[(i\lambda)(ik_{4\nu})(-i)(k_3^\nu + k_4^\nu)\Big] \\ &= -i\lambda^2\ \left(m_2^2 + k_1\cdot k_2\right)^2\ \frac{1}{s - m_3^2 + i\varepsilon}\,, \end{aligned} \tag{3.6.39}$$

where m_1, m_2 and m_3 stands for the masses of the ϕ_1, ϕ_2 and ϕ_3 states. We moreover recall that $k_1 + k_2 = k_3 + k_4$, and that $s = (k_1 + k_2)^2 = (k_3 + k_4)^2$. The signs of the different factors $\pm ik^\mu$ are deduced from the necessity of a creation or annihilation operator at each vertex, which is itself deduced from the momentum flow. In general, all initial-state four-momenta are considered to be incoming (*i.e.* flowing from the left to the right in the diagram above), whereas all final-state four-momenta are considered to be outgoing (*i.e.* also flowing from the left to the right in the diagram above).

Finally, Lagrangian terms involving a total derivative $\partial_\mu(\phi_1 \dots \phi_n)$ do not contribute to any matrix element, as four-momentum has to be conserved at each interaction vertex. Consequently, the different $\pm ik_i$ factors arising from the action of the derivative on each of the fields ϕ_i exactly cancel each other.

Exercise 3.17. Consider the so-called ϕ^4 theory describing the dynamics of a self-interacting real scalar field ϕ of mass m. The associated Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda_4}{4!}\phi^4\,,$$

where the dimensionless parameter λ_4 represents the strength of the scalar self-interactions.

1. Extract the Feynman rules associated with all interaction vertices of the theory.
2. Draw all Feynman diagrams relevant for the process $\phi\phi \to \phi\phi$, and calculate the corresponding matrix element squared $|\mathcal{M}|^2$.
3. Deduce from this result the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$, in which an additional factor of 1/2 must be included to account for the two identical particles in the final state and to prevent some momentum configurations from being counted twice.
4. Derive the total cross section associated with $\phi\phi \to \phi\phi$ scattering in the ϕ^4 theory.

Question 3 of the previous exercise emphasises an important observation valid in the context of scattering processes leading to the production of one or more than one set of n indistinguishable particles.

For final states involving indistinguishable particles, the cross section and decay rate formulas (3.2.76) and (3.3.2) must include an additional symmetry factor of $1/n!$ for each set of n indistinguishable particles. This accounts for the double-counting of some momentum configurations upon phase-space integration.

$$\mathrm{d}\sigma = \frac{1}{S\mathcal{F}} \left|\langle f|\,\mathcal{M}\,|i\rangle\right|^2 \mathrm{dPS}^{(n)} \qquad \text{and} \qquad \mathrm{d}\Gamma = \frac{1}{2SE} \left|\langle f|\,\mathcal{M}\,|i\rangle\right|^2 \mathrm{dPS}^{(n)}\,, \tag{3.6.40}$$

where $\mathcal{F}$ is the Møller flux factor, S the additional symmetry factor, $\mathrm{dPS}^{(n)}$ is the n-body phase space and $\left|\langle f|\,\mathcal{M}\,|i\rangle\right|^2$ the matrix element squared relevant for the process considered. Moreover, in the second formula E stands for the energy of the decaying particle.

Exercise 3.18. Consider the so-called ϕ^3 theory describing the dynamics of a self-interacting real scalar field ϕ of mass m. The associated Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda_3}{3!}\phi^3\,,$$

where the dimensionless parameter λ_3 represents the strength of the scalar self-interactions.

1. Extract the Feynman rules associated with all interaction vertices of the theory.
2. Draw the three Feynman diagrams relevant for the process $\phi\phi \to \phi\phi$. Identify which diagram corresponds to the s-channel contribution, and which diagrams correspond to the t-channel contribution. Justify these names.
3. Calculate the corresponding matrix element $i\mathcal{M}$, as well as the matrix element squared $|\mathcal{M}|^2$.
4. Deduce from this result the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$, writing it in terms of Mandelstam variables.

Exercise 3.19. Consider three scalar fields ϕ_1, ϕ_2 and ϕ_3 whose interaction is driven by the Lagrangian

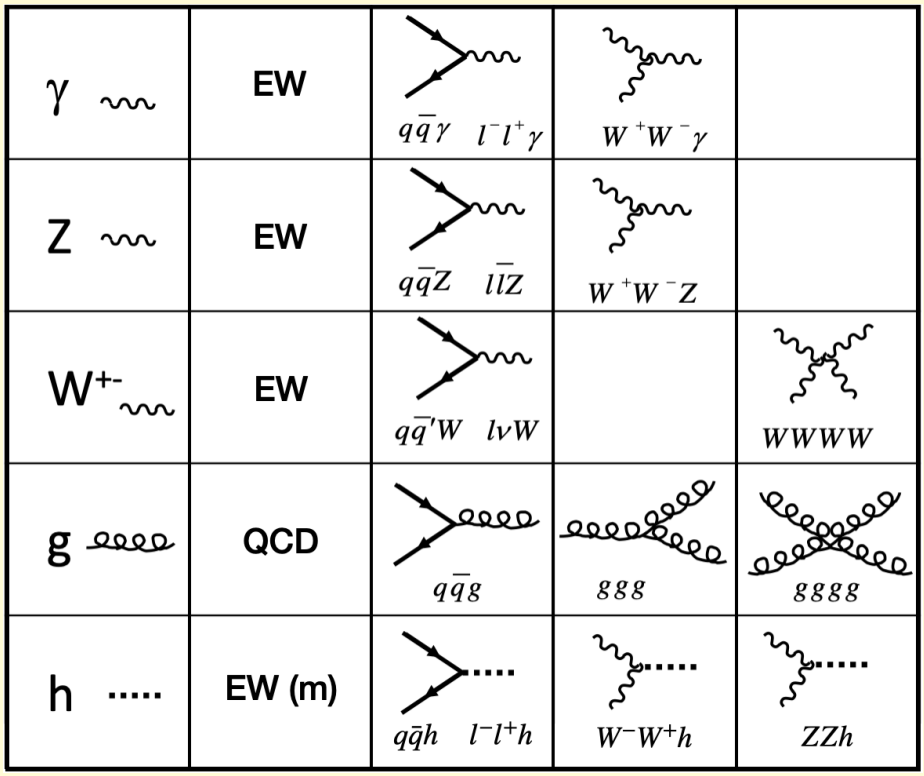
$$\mathcal{L}_{\text{int}} = -\lambda\ \partial_\mu\phi_1\ \partial^\mu\phi_2\ \phi_3 - \lambda\ \phi_1\ \Box\phi_2\ \phi_3\,,$$

where the dimensionless parameter λ represents the strength of all interactions.

1. Extract the Feynman rules associated with all interaction vertices of the theory.
2. Draw the Feynman diagrams relevant for the process $\phi_1\phi_2 \to \phi_1\phi_2$.
3. Calculate the corresponding matrix element $i\mathcal{M}$, as well as the matrix element squared $|\mathcal{M}|^2$.
4. Compare with (3.6.39) and conclude.

Exercise 3.20. In the figure below, we present most of the Feynman rules relevant for the particles of the Standard Model. This list includes:

- ☐ the electroweak (EW) interactions of the quarks (q), antiquarks ($\bar{q}$), leptons (l^-) and antileptons (l^+) with the electroweak gauge bosons (the photon γ and the weak bosons $W^\pm$ and Z);
- ☐ the strong (QCD) interactions of the quarks, antiquarks and gluons (g);
- ☐ the trilinear interactions of the Higgs boson (h) that are of an electroweak nature, but additionally proportional to the mass m of the involved particles.



1. Draw the three Feynman diagrams associated with the process $u\,\bar{u} \to t\,\bar{t}$, in which the symbols u and t stand for an up quark and a top quark respectively, and $\bar{u}$ and $\bar{t}$ are the associated antiquarks.
2. Is one of these three diagrams dominant relative to the others? Justify.
3. Draw the three Feynman diagrams associated with the process $g\,g \to t\,\bar{t}$.
4. Draw all Feynman diagrams associated with the process $g\,g \to t\,\bar{t}\,h$.
5. Draw the dominant Feynman diagrams associated with the process $g\,g \to t\,\bar{t}\,b\,\bar{b}$, the symbols b and $\bar{b}$ referring to a bottom quark and antiquark respectively.

3.7 Summary

Calculations relevant for scattering processes in QFT lie at the heart of the methods that are detailed in these lecture notes. We considered $2 \to n$ processes in which two particles annihilate to produce a final state comprising n particles, and $1 \to n$ processes in which one particle decays into n other particles. These two classes of processes are of particular relevance in light of past, present and future high-energy physics data. In this context, two quantities are especially suitable for a comparison of predictions with measurements. They consist of the differential cross section $\mathrm{d}\sigma$ associated with a $2 \to n$ scattering (the process $ab \to i_1 \dots 1_n$), and the differential decay rate $\mathrm{d}\Gamma$ associated with a $1 \to n$ decay (the process

$a \to i_1 \dots 1_n$).

In this chapter, we first introduced these two notions, not only in QFT but also in non-relativistic quantum mechanics, as far as is concerned with cross sections. On the basis of a simple example, we illustrated that the two definitions match in the non-relativistic limit. Then, we performed a detailed derivation of how to express cross sections and decay rate formulas in QFT in a compact and elegant manner. We demonstrated how this is linked to the notion of an S-matrix, asymptotic states, propagators, the LSZ reduction formula, Wick's theorem, Feynman diagrams and Feynman rules. We hence derived important Lorentz-invariant equations, providing means to calculate scattering rates from the interaction Lagrangian associated with any theory. These formulas are given by

$$\begin{aligned} \mathrm{d}\sigma(ab \to i_1 \dots i_n) &= \frac{1}{S\mathcal{F}} \left|\langle f| \mathcal{M} |i\rangle\right|^2 \mathrm{dPS}^{(n)}\,, \\ \mathrm{d}\Gamma(a \to i_1 \dots i_n) &= \frac{1}{2SE_a} \left|\langle f| \mathcal{M} |i\rangle\right|^2 \mathrm{dPS}^{(n)}\,, \end{aligned} \tag{3.7.1}$$

for a cross section and a decay rate respectively. At their core lies a quantity known as the matrix element squared $|\langle f| \mathcal{M} |i\rangle|^2$ associated with the process considered, with i representing the initial state and f the final state. This last quantity can be deduced in a straightforward manner from Feynman diagrams. The set of relevant Feynman diagrams is extracted as follows:

1. We begin by drawing one end of the diagram for each of the external particles.
2. We then draw a line starting at each of these end.
3. Each line can either contract to an existing line associated with a field of the same nature, or it can split at a new vertex. The form of this splitting (*i.e.* the number of lines emerging from a vertex and their nature in terms of fields) is given by the interaction Lagrangian of the theory, that dictates how fields interact. Moreover, the maximum number of vertices allowed in a diagram is fixed by the perturbative order to which the calculation is achieved.
4. The full amplitude $\langle f| \mathcal{M} |i\rangle$ to a given order in perturbation theory is the sum of all existing diagrams. Only diagrams that are fully connected must be considered, as disconnected diagrams in which each sub-graph is connected to at least one external point can be (and must be) considered independently.

A specific diagram can then be converted into an equation in momentum space as follows:

1. We fix the four-momentum flow along all lines of the diagram in such a way that four-momentum conservation is enforced at each vertex. When there are several diagrams, the chosen momentum flow is the same for all external lines.
2. For each internal line of the diagram, we include a Feynman propagator in Fourier space $\frac{i}{p^2 - m^2 + i\varepsilon}$, where p stands for the four-momentum flowing through the internal line.
3. Lines connected to external points are replaced by 1.
4. The contributions associated with each vertex are obtained from the prefactors of the different terms of the interaction Lagrangian, together with four-momentum-dependent factors in case of derivative couplings.
5. Integration over all undetermined (internal) four-momenta must be performed.
6. Each contribution must be divided by the appropriate symmetry factor, depending on the geometry of the diagram.

The obtained quantity must finally be squared, and the scattering rates are then obtained by multiplying this matrix element squared by the n-body phase space $\mathrm{dPS}^{(n)}$, the inverse of the Møller flux factor $\mathcal{F}$ for a scattering process or that of the initial-state particle energy E_a for a decay process, and a symmetry factor S. The phase space is given by

$$\mathrm{dPS}^{(n)} = (2\pi)^4\, \delta^{(4)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big) \prod_{i=1}^{n} \left[\frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_i}\right], \tag{3.7.2}$$

where p_a^μ and p_b^μ are the four-momenta of the initial-state particles, and p_i^μ, $\mathbf{p}_i$ and E_i stand for the four-momentum, the momentum and the energy of the i^{th} final-state particle respectively. The Møller flux factor $\mathcal{F}$ is defined by

$$\mathcal{F} = 4\sqrt{\left(p_a \cdot p_b\right)^2 - m_a^2 m_b^2}\,, \tag{3.7.3}$$

where m_a and m_b are the masses of the initial-state particles. Finally, the symmetry factor S contains a factor of $n!$ for each set of n indistinguishable particles in the final state.

Bibliography

[1] H. Minkowski, *Raum und Zeit*, *Physikalische Zeitschrift* **10** (1909) 75–88.

[2] A. Einstein, *Zur Elektrodynamik bewegter Körper*, *Annalen der Physik* **322** (1905) 891–921.

[3] H. A. Lorentz, *Simplified Theory of Electrical and Optical Phenomena in Moving Systems*, *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* **1** (1898) 427–442.

[4] H. A. Lorentz, *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*, *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* **6** (1903) 809–831.

[5] H. M. Poincaré, *Sur la dynamique de l'électron*, *Rendiconti del Circolo Matematico di Palermo* **21** (1906) 129–176.

[6] E. P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, *Annals Math.* **40** (1939) 149–204.

[7] V. Bargmann and E. P. Wigner, *Group Theoretical Discussion of Relativistic Wave Equations*, *Proc. Nat. Acad. Sci.* **34** (1948) 211.

[8] J. Lubański, *Sur la theorie des particules élémentaires de spin quelconque I*, *Physica* **9** (1942) 310–324.

[9] M. Planck, *Ueber das gesetz der energieverteilung im normalspectrum*, *Annalen der Physik* **309** (1901) 553–563.

[10] A. Einstein, *Über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt*, *Annalen der Physik* **322** (1905) 132–148.

[11] E. Schrödinger, *An undulatory theory of the mechanics of atoms and molecules*, *Phys. Rev.* **28** (1926) 1049–1070.

[12] P. A. M. Dirac, *A new notation for quantum mechanics*, *Mathematical Proceedings of the Cambridge Philosophical Society* **35** (1939) 416–418.

[13] J. L. Lagrange, *Analytical Mechanics*, vol. 191 of *Boston Studies in the Philosophy and History of Science*. Springer Dordrecht, 2010, 10.1007/978-94-015-8903-1.

[14] W. R. Hamilton, *On a General Method of Expressing the Paths of Light & of the Planets by the Coefficients of a Characteristic Function*. P.D. Hardy, Dublin, 1833.

[15] E. Noether, *Invariante variationsprobleme*, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1918) 235–257.

[16] O. Klein, *Quantentheorie und fünfdimensionale relativitätstheorie*, *Zeitschrift für Physik* **37** (1926) 895–906.

[17] W. Gordon, *Der comptoneffekt nach der schrödingerschen theorie*, *Zeitschrift für Physik* **40** (1926) 117–133.

[18] P. A. M. Dirac, *The quantum theory of the emission and absorption of radiation*, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **114** (1927) 243–265.

[19] R. P. Feynman, *The Theory of positrons*, *Phys. Rev.* **76** (1949) 749–759.

[20] H. Lehmann, K. Symanzik and W. Zimmermann, *On the formulation of quantized field theories*, *Nuovo Cim.* **1** (1955) 205–225.

[21] G. C. Wick, *The Evaluation of the Collision Matrix*, *Phys. Rev.* **80** (1950) 268–272.

[22] H. Geiger and E. Marsden, *On a diffuse reflection of the α-particles*, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **82** (1909) 495–500.

[23] H. Geiger, *The scattering of α-particles by matter*, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **83** (1910) 492–504.

[24] E. Rutherford, *The scattering of alpha and beta particles by matter and the structure of the atom*, *Phil. Mag. Ser. 6* **21** (1911) 669–688.

[25] S. Mandelstam, *Determination of the pion - nucleon scattering amplitude from dispersion relations and unitarity. General theory*, *Phys. Rev.* **112** (1958) 1344–1360.

[26] G. Goldberg, *A Rule for the Combinatoric Factors of Feynman Diagrams*, *Phys. Rev. D* **32** (1985) 3331.

[27] E. H. Wichmann and J. H. Crichton, *Cluster Decomposition Properties of the S Matrix*, *Phys. Rev.* **132** (1963) 2788–2799.

[28] S. Weinberg, *What is quantum field theory, and what did we think it is?*, in *Conference on Historical Examination and Philosophical Reflections on the Foundations of Quantum Field Theory*, pp. 241–251, 3, 1996. hep-th/9702027.