1 Symétries du lagrangien de Dirac (*)

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$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

1.1 $\psi \to e^{i\theta}\psi, \quad \bar{\psi} \to e^{-i\theta}\bar{\psi}$

$$\mathcal{L} \longrightarrow \bar{\psi}\, e^{-i\theta}(i\gamma^\mu \partial_\mu - m)\, e^{i\theta}\psi$$

$$= \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi = \mathcal{L}$$

est invariante

$$\delta\theta\, J^\mu_V = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi}\,\delta\psi + \delta\bar{\psi}\,\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}}$$

transformation infinitesimale : $\psi' = e^{i\delta\theta}\psi = \psi + i\delta\theta\psi$

$$\delta\psi = i\,\delta\theta\,\psi \qquad \delta\bar{\psi} = -i\,\delta\theta\,\bar{\psi}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\,\bar{\psi}\gamma^\mu$$

$$\Rightarrow \quad \boxed{J^\mu_V = -\bar{\psi}\gamma^\mu\psi}$$

1.2
1

$$\{\gamma^\mu, \gamma_5\} = 0 \Rightarrow \gamma^\mu e^{i\xi\gamma_5} = e^{-i\xi\gamma_5}\gamma^\mu$$

$$\Rightarrow \bar{\psi}\psi \longrightarrow \psi^\dagger e^{-i\xi\gamma_5}\gamma^0 e^{i\xi\gamma_5}\psi = \bar{\psi}\, e^{2i\xi\gamma_5}\psi$$

$$\bar{\psi}\gamma^\mu\psi \to \psi^\dagger e^{-i\xi\gamma_5}\gamma^0\gamma^\mu e^{i\xi\gamma_5}\psi = \bar{\psi}\gamma^\mu\psi$$

$$\Rightarrow \mathcal{L} \longrightarrow \bar{\psi}\, i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\, e^{2i\xi\gamma_5}\psi \neq \mathcal{L}$$

si $m \neq 0$

"Théorème" de Nöther pour une transformation qui n'est pas une symétrie :

$$0 \neq \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\mu \delta\varphi + \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\mu \delta\varphi + \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi \right) = \left(\partial_\mu J^\mu \right) \delta\alpha$$

$$\Rightarrow \quad \boxed{\partial_\mu J^\mu = \frac{\delta \mathcal{L}}{\delta \alpha} \neq 0}$$

3. transformation infinitésimale :

$$\psi \to e^{i\xi\gamma_5}\psi \simeq \psi + i\xi\gamma_5\psi$$

$$\Rightarrow \quad \delta\psi = i\xi\gamma_5\psi$$

$$\delta\bar\psi = (\delta\psi)^\dagger\gamma^0 = -i\xi\psi^\dagger\gamma_5\gamma_0 = i\xi\bar\psi\gamma_5$$

$$\xi J^\mu_A = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi = i\bar\psi\gamma^\mu(i\xi\gamma_5\psi)$$

$$\Rightarrow \boxed{J^\mu_A = -\bar\psi\gamma^\mu\gamma_5\psi}$$

$$\partial^\mu_\mu J^\mu_A = -(\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma_5\gamma^\mu\partial_\mu\psi$$

$$\begin{aligned} i(\partial_\mu\bar\psi)\gamma^\mu &= -m\bar\psi \\ i\gamma^\mu\partial_\mu\psi &= m\psi \end{aligned} \Rightarrow \boxed{\partial^\mu J_{\mu A} = -2im\bar\psi\gamma^5\psi}$$

Sous une transformation infinitésimale :

$$\frac{\delta\mathcal{L}}{\delta\xi} = \frac{\delta(-m\bar\psi\, e^{2i\delta\xi\gamma_5}\psi)}{\delta\xi} = -2im\bar\psi\gamma_5\psi$$

$$\Rightarrow \partial^\mu J_{\mu A} = +\frac{\delta\mathcal{L}}{\delta\xi}$$ en accord avec le point précédent.

1.3

1. $\Psi \to e^{i\theta}\Psi \Rightarrow \begin{cases} \psi_L \to e^{i\theta}\psi_L \\ \psi_R \to e^{+i\theta}\psi_R \end{cases}$ $\quad U(1)_V$

$\psi \to e^{i\xi\gamma_5}\psi \Rightarrow \begin{cases} \psi_L \to e^{i\xi}\psi_L \\ \psi_R \to e^{-i\xi}\psi_R \end{cases}$ $\quad U(1)_A$

on peut prendre une transformation avec

- $\theta = \xi \equiv \alpha_L$: $\begin{cases} \psi_L \to e^{i\alpha_L}\psi_L \\ \psi_R \to \psi_R \end{cases}$ $\quad U(1)_L$

- $\theta = -\xi = \alpha_R$ $\begin{cases} \psi_L \to \psi_L \\ \psi_R \to e^{i\alpha_R}\psi_R \end{cases}$ $\quad U(1)_R$

2.

pour ~~$m \neq 0$~~ $m=0$ le lagrangien a une symétrie $U(1)_L \times U(1)_R$ car :

$$L = \underbrace{i\bar{\Psi}_L \not{\partial} \psi_L}_{\text{Invariante sous } U(1)_L} + \underbrace{i\bar{\Psi}_R \not{\partial} \psi_R}_{\text{Invariante sous } U(1)_R}$$