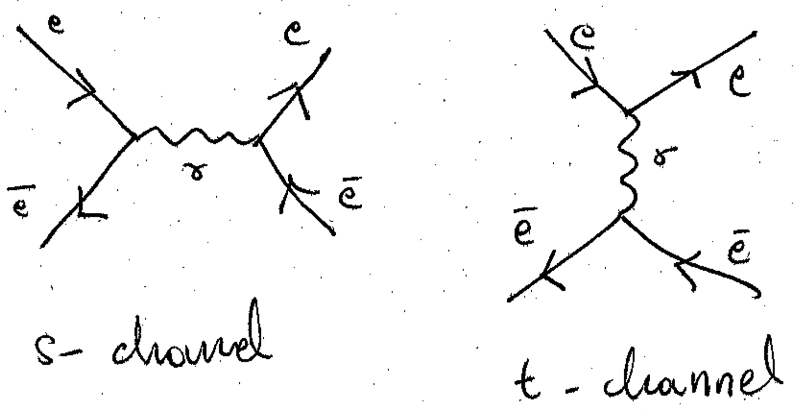
QFT - TD 4

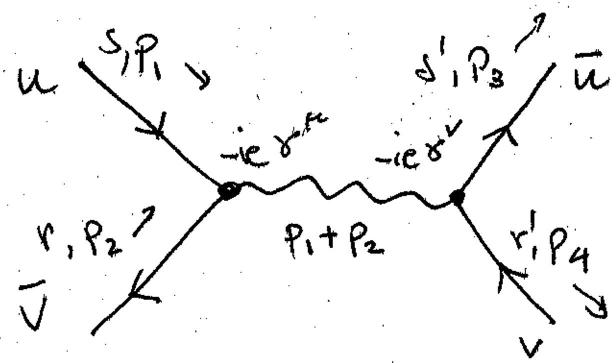
2. QED Cross-sections

2.1 $e^+e^- \longrightarrow e^+e^-$ (Bhabha, 1936)

1. At tree-level we have the two diagrams:

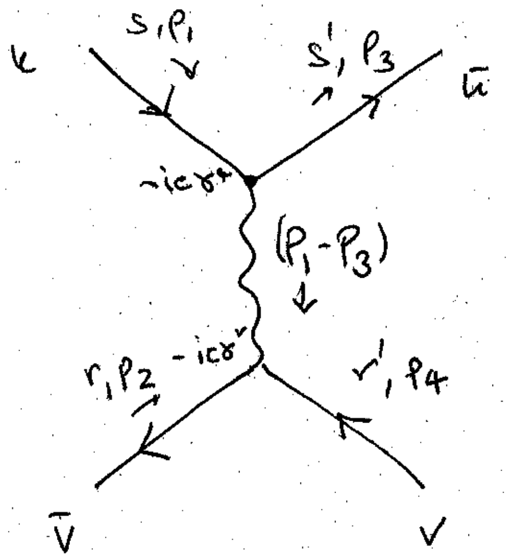


2.



s- channel Amplitude

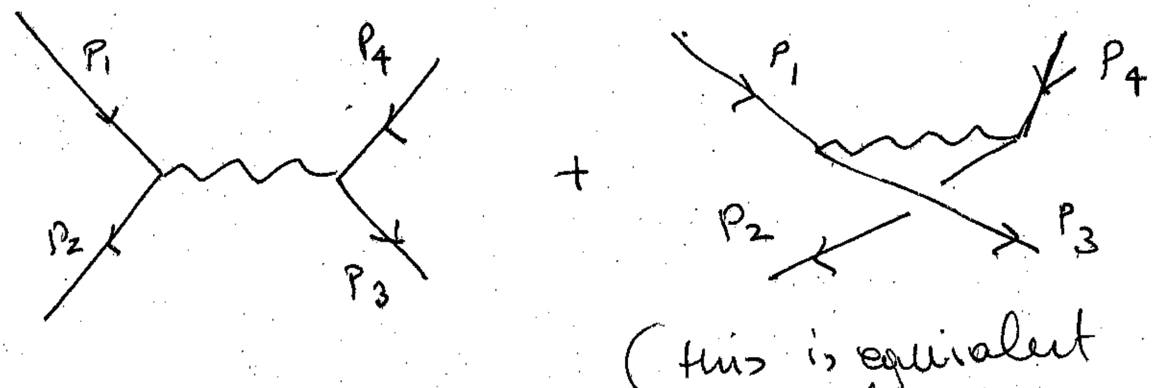
$$\mathcal{A}_s = (-i)\,\bar{v}^r(p_2)(-ie\gamma^\mu)u^s(p_1)\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\times$$
$$\bar{u}^{s'}(p_3)(-ie\gamma^\nu)v^{r'}(p_4)$$



t - channel
Amplitude

$$\mathcal{A}_t = (-i)\, \bar{v}^r(p_2)(-ie\gamma^\nu)v^{r'}(p_4)\left(-\frac{i\, g_{\mu\nu}}{(p_1-p_3)^2}\right) \times \bar{u}^{s'}(p_3)(-ie\gamma^\mu)u^s(p_1)$$

The two terms have a relative $-$ sign due to Fermi statistics: in fact, we can show the two diagrams in the following way:



(this is equivalent to the t-channel diagram, since p_1 is connected to p_3 and p_2 to p_4).

Clearly they are related by exchange of one fermionic leg $\Rightarrow$ $-$ sign!

$$\mathcal{A} = \mathcal{A}_t + \mathcal{A}_s = (-i)(-ie)^2(-i) \times$$

$$\left[\frac{\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\bar{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)}{(p_1+p_2)^2} - \frac{\bar{v}^r(p_2)\gamma^\mu v^{r'}(p_4)\bar{u}^{s'}(p_3)\gamma_\mu u^s(p_1)}{(p_1-p_3)^2} \right]$$

introduce $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$

$$\mathcal{A} = e^2 \left[\frac{1}{s} \bar{v}^r(p_2)\gamma^\mu u^s(p_1)\bar{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4) - \frac{1}{t} \bar{v}^r(p_2)\gamma^\mu v^{r'}(p_4)\bar{u}^{s'}(p_3)\gamma_\mu u^s(p_1) \right]$$

3. We want to compute

$$\frac{1}{4}\sum_{\text{spins}}' |\mathcal{A}|^2 = \frac{1}{4} \sum_{\substack{s,s' \\ r,r'}} \left(|\mathcal{A}_s|^2 + |\mathcal{A}_t|^2 + \mathcal{A}_s^\dagger \mathcal{A}_t + \mathcal{A}_t^\dagger \mathcal{A}_s \right)$$

where:
$$\mathcal{A}_s = \frac{e^2}{s} \bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu v(p_4)$$
$$\mathcal{A}_t = -\frac{e^2}{t} \bar{v}(p_2)\gamma^\mu v(p_4)\bar{u}(p_3)\gamma_\mu u(p_1)$$

$$A_s^{\dagger} A_t = -\frac{e^4}{st} \left\{ \begin{array}{l} \left(\bar{u}(p_1)\gamma^{\mu} v(p_2)\right)\left(\bar{v}(p_4)\gamma_{\mu} u(p_3)\right) \\ \left(\bar{v}(p_2)\gamma^{\nu} v(p_4)\right)\left(\bar{u}(p_3)\gamma_{\nu} u(p_1)\right) \end{array} \right.$$

$$\to -\frac{e^4}{4st}\left[\bar{u}^{s}_{\alpha}(p_1)\gamma^{\mu}_{\alpha\beta} v^{r}_{\beta}(p_2)\, \bar{v}^{r}_{\lambda}(p_2)\gamma^{\nu}_{\lambda\delta} v^{r'}_{\delta}(p_4)\, \bar{v}^{r'}_{\epsilon}(p_4)\gamma_{\mu\,\epsilon\sigma} u^{s'}_{\sigma}(p_3) \right.$$
$$\left. \bar{u}^{s'}_{\xi}(p_3)\gamma_{\nu\,\xi\eta} u^{s}_{\eta}(p_1) \right]$$

$$= \frac{-e^4}{4st}\left[(\not{p}_1 + m)_{\eta\alpha}\gamma^{\mu}_{\alpha\beta}(\not{p}_2 - m)_{\beta\lambda}\gamma^{\nu}_{\lambda\delta}(\not{p}_4 - m)_{\delta\epsilon} \right.$$
$$\left. \gamma_{\mu\,\epsilon\sigma}(\not{p}_3 + m)_{\sigma\xi}\gamma_{\nu\,\xi\eta} \right]$$

$$= -\frac{e^4}{4st}\,\mathrm{Tr}\left[(\not{p}_1 + m)\gamma^{\mu}(\not{p}_2 - m)\gamma^{\nu}(\not{p}_4 - m)\gamma_{\mu} \right.$$
$$\left. (\not{p}_3 + m)\gamma_{\nu} \right]$$

so now it is a single trace

~~now:~~

$$\cancel{\gamma^{\mu}(\not{p}_2 - m)\gamma^{\nu} = -\gamma^{\mu}\gamma^{\nu}(\not{p}_2 - m)}$$
$$\cancel{+ \gamma^{\mu}\{\not{p}_2 - m, \gamma^{\nu}}$$

let us consider each of the four terms:

$$|A_s|^2 = \frac{e^4}{s^2}\left(\bar{v}^{r'}(p_4)\gamma_\mu u^{s'}(p_3)\right)\left(\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right)$$
$$\left(\bar{u}^{s}(p_1)\gamma^\mu v^{r}(p_2)\right)\left(\bar{v}^{r}(p_2)\gamma^\nu u^{s}(p_1)\right)$$

Using the spin sum-rules:

$$\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta} \qquad \left(\not{p}_{\alpha\beta} \equiv \gamma^\mu_{\alpha\beta} p_\mu\right)$$

$$\sum_s v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

and ordering the 4 factors above in a cyclic fashion, we find: (recall $\not{p} \equiv \gamma^\mu p_\mu$)

$$\frac{1}{4}\sum_{\substack{s,s'\\ r,r'}}|A_s|^2 = \frac{e^4}{4s^2}\,\mathrm{Tr}\left[(\not{p}_4 - m)\gamma_\mu(\not{p}_3 + m)\gamma_\nu\right]\cdot$$
$$\mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu\right]$$

Now we use the trace identities:

$$\mathrm{Tr}\,(\text{odd-number of } \gamma\text{'s}) = 0$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

So :

$$\mathrm{Tr}[(\not{p}_4 - m)\gamma_\mu(\not{p}_3 + m)\gamma_\nu] = \mathrm{Tr}(\not{p}_4\gamma_\mu\not{p}_3\gamma_\nu) - m^2\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu$$

$$= p_4^\rho p_3^\sigma\,\mathrm{Tr}(\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu) - m^2\,\mathrm{Tr}(\gamma_\mu\gamma_\nu) =$$

$$= 4\,p_4^\rho p_3^\sigma\left[g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right] - 4m^2 g_{\mu\nu}$$

$$= 4\left(p_{4\mu}p_{3\nu} + p_{4\nu}p_{3\mu} - p_4\cdot p_3\,g_{\mu\nu} - m^2 g_{\mu\nu}\right)$$

Similarly :

$$\mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu] =$$

$$= 4\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1\cdot p_2\,g^{\mu\nu} - m^2 g^{\mu\nu}\right)$$

Contracting the two ~~to~~ tensors we find:

$$\frac{1}{4}\sum_{\text{spins}}|A_s|^2 = \frac{e^4}{4s^2}\cdot 16\left(p_{4\mu}p_{3\nu} + p_{4\nu}p_{3\mu} - (m^2 + p_3\cdot p_4)g_{\mu\nu}\right)\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (m^2 + p_1 p_2)g^{\mu\nu}\right)$$

call: $p_{ij} \equiv p_i^\mu \cdot p_{j\mu}$

$$= \frac{4e^4}{s^2}\left\{2p_{14}p_{23} + 2p_{24}p_{13} - 2(m^2 + p_{12})p_{34} - 2(m^2 + p_{34})p_{12} + 4(m^2 + p_{12})(m^2 + p_{34})\right\}$$

$$= \frac{4e^4}{s^2}\left\{2p_{14}p_{23} + 2p_{24}p_{13} + 2m^2 p_{34} + 2m^2 p_{12} + 4m^4\right\}$$

Finally:

$$\frac{1}{4}\sum_{\text{spins}} |A_s|^2 = \frac{8e^4}{s^2}\left[p_{14}p_{23} + p_{13}p_{24} + m^2 p_{12} + m^2 p_{34} + 2m^4\right]$$

(this is the same as in $e^+e^- \to \mu^+\mu^-$ except that here the masses are all equal).

Now consider the $|A_t|^2$ term:

$$|A_t|^2 = \frac{e^4}{t^2}\left(\bar{v}^{r'}(p_4)\gamma^\mu v^r(p_2)\right)\left(\bar{v}^r(p_2)\gamma^\nu v^{r'}(p_4)\right)$$
$$\left(\bar{u}^s(p_1)\gamma_\mu u^{s'}(p_3)\right)\left(\bar{u}^{s'}(p_3)\gamma_\nu u^s(p_1)\right)$$

Summing over polarisations:

$$\frac{1}{4}\sum_{\text{spins}} |A_t|^2 = \frac{e^4}{4t^2}\,\mathrm{Tr}\left[(\not{p}_2 - m)\gamma^\nu(\not{p}_4 - m)\gamma^\mu\right]$$
$$\mathrm{Tr}\left[(\not{p}_3 + m)\gamma_\nu(\not{p}_1 + m)\gamma_\mu\right]$$

$$= \frac{e^4}{4t^2}\left[\mathrm{Tr}[\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu] + m^2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu)\right]$$
$$\left[\mathrm{Tr}[\not{p}_3\gamma_\nu\not{p}_1\gamma_\mu] + m^2\,\mathrm{Tr}(\gamma_\mu\gamma_\nu)\right]$$

$$= \frac{4e^4}{t^2}\left(p_{2\nu}p_{4\mu} + p_{2\mu}p_{4\nu} + (m^2 - p_2p_4)g_{\mu\nu}\right)$$
$$\left(p_{3\nu}p_{1\mu} + p_{3\mu}p_{1\nu} + (m^2 - p_1p_3)g_{\mu\nu}\right) =$$

$$= \frac{8e^4}{t^2}\left(p_{23}p_{14} + p_{12}p_{34} - m^2(p_{24} + p_{13}) + 2m^4\right)$$

Cross terms

take first : $\frac{1}{4}\sum_{spins}' A_t^\dagger A_s$

$$A_t^\dagger A_s = -\frac{e^4}{st}\left(\bar{v}^{r'}(p_4)\gamma^\mu v^r(p_2)\right)\left(\bar{u}^s(p_1)\gamma_\mu u^{s'}(p_3)\right)$$
$$\left(\bar{u}^{s'}(p_3)\gamma^\nu v^{r'}(p_4)\right)\left(\bar{v}^r(p_2)\gamma_\nu u^s(p_1)\right)$$

reshuffle:

$$= -\frac{e^4}{st}\Big[\bar{v}^r(p_2)\gamma_\nu u^s(p_1)\bar{u}^s(p_1)\gamma_\mu u^{s'}(p_3)\bar{u}^{s'}(p_3)\gamma^\nu v^{r'}(p_4)\times$$
$$\bar{v}^{r'}(p_4)\gamma^\mu v^r(p_2)\Big]$$

Now summing over r, s, r', s' gives a single trace:

$$\frac{1}{4}\sum_{\substack{r,r'\\ s,s'}}' A_t^\dagger A_s = -\frac{e^4}{4st}\times$$

$$\times \mathrm{Tr}\left((\not{p}_1+m)\gamma_\nu(\not{p}_2-m)\gamma_\mu(\not{p}_4-m)\gamma^\nu(\not{p}_3+m)\gamma^\mu\right]$$

this gives trace of 8, 6 and 4 gamma matrices.

Simplification: go to the high-energy limit, where we can ignore the electron mass: $E_{CM} \gg m$.

then the most important term is the one which does not contain any m:

$$\frac{1}{4}\sum' A_t^\dagger A_s \simeq -\frac{e^4}{4st}\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\not{p}_3\gamma^\mu\right)$$

so let us compute this thing:

We want to bring two $\gamma^\mu \cdots \gamma_\mu \cdots$ close so we can contract them and use:

$$\gamma^\mu\gamma_\mu = g^{\mu\nu}\gamma_\mu\gamma_\nu = \frac{1}{2}g^{\mu\nu}\{\gamma_\mu,\gamma_\nu\}$$
$$= g^{\mu\nu}g_{\mu\nu}\mathbb{1} = 4\mathbb{1}.$$

Also, notice that: $\{\gamma_\mu, \not{p}\} = 2p_\mu$

and $$\gamma^\mu\not{p}\gamma_\mu = -\underbrace{\gamma^\mu\gamma_\mu\not{p}}_{4\not{p}} + \underbrace{\gamma^\mu 2p_\mu}_{2\not{p}}$$
$$= -2\not{p}$$

So:

$$\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\not{p}_3\gamma^\mu\right) = -\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\gamma^\mu\not{p}_3\right)$$
$$+ 2p_3^\mu\,\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\right)$$
$$= +\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\mu\gamma^\nu\not{p}_3\right) - 2g^{\mu\nu}\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\not{p}_3\right)$$
$$+ 2\,\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\not{p}_3\not{p}_4\gamma^\nu\right) =$$
$$= -2\,\mathrm{Tr}\left(\not{p}_1\gamma_\nu\not{p}_2\not{p}_4\gamma^\nu\not{p}_3\right) - 2\,\mathrm{Tr}\left(\not{p}_1\gamma^\mu\not{p}_2\gamma_\mu\not{p}_4\not{p}_3\right)$$
$$+ 2\,\mathrm{Tr}\left(\gamma^\nu\not{p}_1\gamma_\nu\not{p}_2\not{p}_3\not{p}_4\right) =$$

$$= -2\,\mathrm{Tr}(\not{p}_1 \gamma_\nu \not{p}_2 \{\not{p}_4, \gamma^\nu\} \not{p}_3)$$

$$- 4\,\mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) \qquad \{\not{p}_4, \gamma^\nu\} = 2p_4^\nu$$

$$= -4\,\mathrm{Tr}(\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_3) - 4\,\mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4)$$

$$= -4\left[\mathrm{Tr}(\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_3 + \not{p}_4 \not{p}_1 \not{p}_2 \not{p}_3)\right]$$

$$= -4\,\mathrm{Tr}(\{\not{p}_1, \not{p}_4\} \not{p}_2 \not{p}_3) \qquad \{\not{p}, \not{q}\} = 2\,p \cdot q$$

$$= -8\,P_{14}\,\mathrm{Tr}(\not{p}_2 \not{p}_3) = -32\,P_{14} P_{23}$$

so, in the massless limit:

$$\frac{1}{4}\sum_{\text{spins}}{}' \mathcal{A}_t^\dagger \mathcal{A}_s = \frac{8e^4}{st} P_{14} P_{23}$$

similarly:

$$\frac{1}{4}\sum{}' \mathcal{A}_s^\dagger \mathcal{A}_t = \frac{8e^4}{st} P_{14} \cdot P_{23}$$

Overall, in the massless limit:

$$\frac{1}{4}\sum_{\text{spins}}|A|^2 = \frac{1}{4}\Big[\sum |A_s|^2 + \sum |A_t|^2 + \sum A_s^\dagger A_t + \sum A_t^\dagger A_s\Big]$$

$$= 8e^4\left[\frac{P_{14}P_{23} + P_{13}P_{24}}{S^2} + \frac{P_{14}P_{23} + P_{12}P_{34}}{t^2} + 2\,\frac{P_{14}P_{23}}{tS}\right]$$

$$\cancel{+\frac{P_{14}P_{23}}{tS}}$$

Now recall that:

$$S = (P_1+P_2)^2 = (P_3+P_4)^2 = 2m^2 + 2P_{12} = 2m^2 + 2P_{34}$$

$$\simeq 2P_{12} \simeq 2P_{34} \quad (\text{in the } m^2 \to 0 \text{ limit})$$

Similarly:

$$t = (P_1-P_3)^2 = (P_1-P_4)^2 \simeq -2P_{13} = -2P_{14}$$

$$u = (P_1-P_4)^2 = (P_2-P_3)^2 \simeq -2P_{14} \simeq -2P_{23}$$

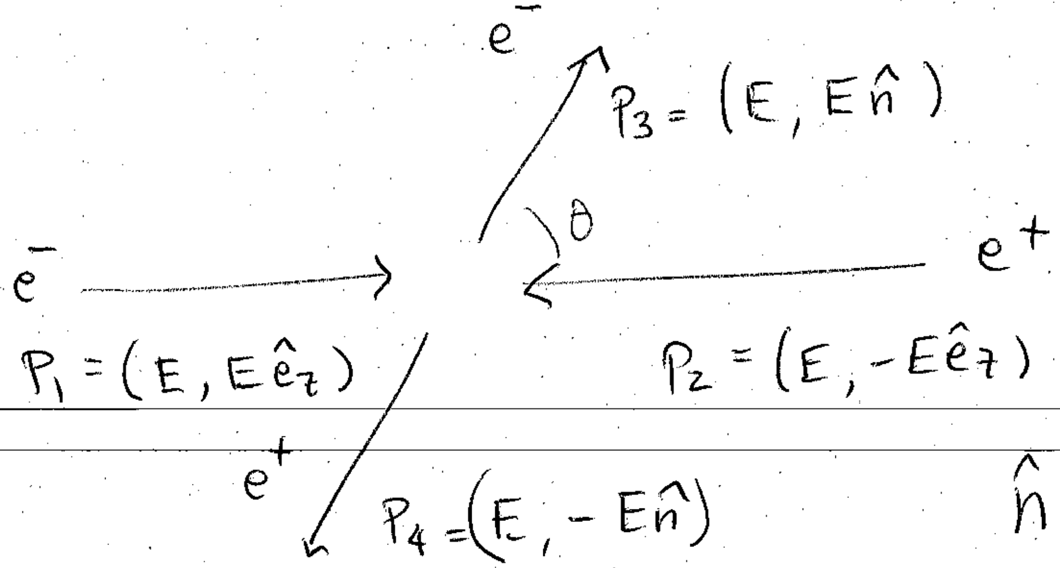
So the parenthesys is:

$$\left[\frac{u^2+t^2}{S^2} + \frac{u^2+S^2}{t^2} + \frac{2u^2}{tS}\right]$$

and finally:

$$\frac{1}{4}\sum_{\text{spins}}{}'|\mathcal{A}|^2 = 8e^4\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2+\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2\right]$$

Now let us express this in terms of E and $\cos\theta$ in the CM frame:



$E_{CM} = 2E$

$\hat{n}\cdot\hat{e}_z = \cos\theta$

$$s = (P_1+P_2)^2 = 2P_1P_2 = 4E^2$$
$$t = (P_1-P_3)^2 = -2P_1P_3 = -2E^2(1-\cos\theta)$$
$$u = (P_1-P_4)^2 = -2P_1P_4 = -2E^2(1+\cos\theta)$$

so:

$$\frac{1}{4}\sum_{\text{spins}}{}'|\mathcal{A}|^2 = 8e^4\left[(1+\cos\theta)^2\left(\frac{1}{2}-\frac{1}{(1-\cos\theta)}\right)^2+\frac{(1-\cos\theta)^2}{4} + \frac{4}{(1-\cos\theta)^2}\right]$$

Finally the cross section is:

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{e^4}{8\pi^2}\frac{1}{E_{CM}}\left[(1+\cos\theta)^2\left(\frac{1}{2}-\frac{1}{1-\cos\theta}\right)^2 + \right.$$
$$\left. + \frac{(1-\cos\theta)^2}{4} + \frac{4}{(1-\cos\theta)^2}\right]$$