1] Fermion Propagator

1) $$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \sum_{ss'}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec p}2\omega_{\vec q}}}$$

$$\langle 0|\left(b^s_{\vec p}\, u^s_\alpha(p)e^{-ip\cdot x} + c^{s\dagger}_{\vec p}\, V^s_\alpha(p)e^{ipx}\right)\cdot$$
$$\left(\bar{u}^{s'}_\beta(q)\, b^{\dagger s'}_{\vec q}\, e^{iqy} + c^{s'}_{\vec q}\, V^{s'}(q)e^{-iqy}\right)|0\rangle =$$

$$\langle 0|\, c^\dagger_{\vec p} = 0 \quad ; \quad c_{\vec q}|0\rangle = 0$$

$$= \sum_{ss'}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{e^{-ipx+iqy}}{\sqrt{2\omega_{\vec p}2\omega_{\vec q}}}\, u^s_\alpha(p)\bar{u}^{s'}_\beta(q)\underbrace{\langle 0|b^s_{\vec p}\, b^{\dagger s'}_{\vec q}|0\rangle}_{(2\pi)^3\delta^{ss'}\delta^{(3)}(\vec p-\vec q)}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2\omega_{\vec p}}\, u^s_\alpha(p)\bar{u}^s_\beta(p) \longleftarrow \text{use spin sum rule}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\not{p}_{\alpha\beta} + m\delta_{\alpha\beta}\right)\frac{e^{-ip(x-y)}}{2\omega_{\vec p}}$$

$$= \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} + m\delta_{\alpha\beta}\right)\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2\omega_{\vec p}}}_{D(x-y)}$$

similarly:

$$\langle 0|\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle = \sum_{ss'}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec p}\,2\omega_{\vec q}}}\cdot$$

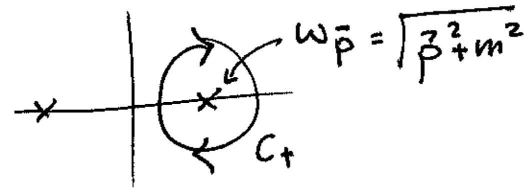
$$e^{-iqy+ipx}\,\bar{V}^s_\beta(q)V^{s'}_\alpha(p)\underbrace{\langle 0|c^s_{\vec q}\,c^{s'\dagger}_{\vec p}|0\rangle}_{(2\pi)^3\delta^{ss'}\delta^3(\vec q-\vec p)}$$

$$= \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{e^{-ip(y-x)}}{2\omega_{\vec p}}\,\underbrace{V^s_\alpha(p)\bar{V}^s_\beta(p)}_{\text{sum rule}}$$

$$= \int\frac{d^3p}{(2\pi)^3}\left(\not{p}_{\alpha\beta}-m\,\delta_{\alpha\beta}\right)\frac{e^{-ip(y-x)}}{2\omega_{\vec p}}$$

$$= \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu}-m\,\delta_{\alpha\beta}\right)\underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{e^{-ip(y-x)}}{2\omega_{\vec p}}}_{D(y-x)}$$

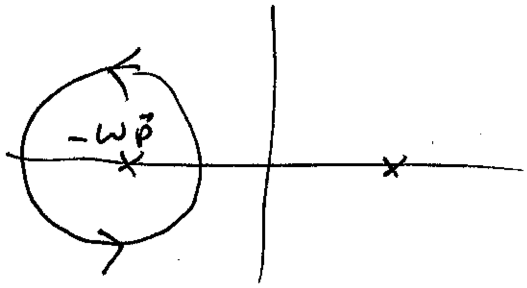
2) Choose C_+:



$$\int_{C_+}\frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\frac{ie^{-ip(x-y)}}{p^2-m^2} = \int\frac{d^3p}{(2\pi)^3}\int\frac{d\omega}{2\pi}\frac{ie^{-ip(x-y)}}{(\omega-\omega_{\vec p})(\omega+\omega_{\vec p})}$$

$$\underset{\text{Cauchy's theorem}}{=}\frac{(-2\pi i)}{2\pi}i\int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\omega_p(x^0-y^0)+\vec p\cdot(\vec x-\vec y)}}{2\omega_{\vec p}} = D(x-y)$$

Choose C_- :



$$\int_{C_-}\frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\,\frac{i\,e^{-ip(x-y)}}{p^2-m^2} = \int\frac{d^3p}{(2\pi)^3}\int_{C_-}\frac{d\omega}{2\pi}\,\frac{i\,e^{-ip(x-y)}}{(\omega-\omega_p)(\omega+\omega_p)}$$

$$= \frac{(2\pi i)}{2\pi}\, i\int\frac{d^3p}{(2\pi)^3}\,\frac{e^{i\omega_{\vec p}(x^0-y^0)+i\vec p(\vec x-\vec y)}}{-2\omega_{\vec p}}$$

change integration variable: $\vec p \to -\vec p$

$$= \int\frac{d^3p}{(2\pi)^3}\,\frac{e^{i(\omega_p(x^0-y^0)-\vec p\cdot(\vec x-\vec y))}}{2\omega_{\vec p}} = D(y-x)$$

3) $\langle 0|T\,\psi_\alpha(x)\,\bar\psi_\beta(y)|0\rangle =$

$$= \theta(x^0-y^0)\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle$$

$$= \theta(x^0-y^0)\left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}+m\delta_{\alpha\beta}\right)D(x-y) - \theta(y^0-x^0)\left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu}-m\delta_{\alpha\beta}\right)D(y-x)$$

$$= \theta(x^0-y^0)\left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}+m\delta_{\alpha\beta}\right)\int\frac{d^3p}{(2\pi)^3}\int_{C_+}\frac{d\omega}{2\pi}\,\frac{i\,e^{-ip(x-y)}}{p^2-m^2}$$

$$- \theta(y^0-x^0)\left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu}-m\delta_{\alpha\beta}\right)\int\frac{d^3p}{(2\pi)^3}\int_{C_-}\frac{d\omega}{2\pi}\,\frac{i\,e^{-ip(x-y)}}{p^2-m^2}$$

$$\downarrow$$

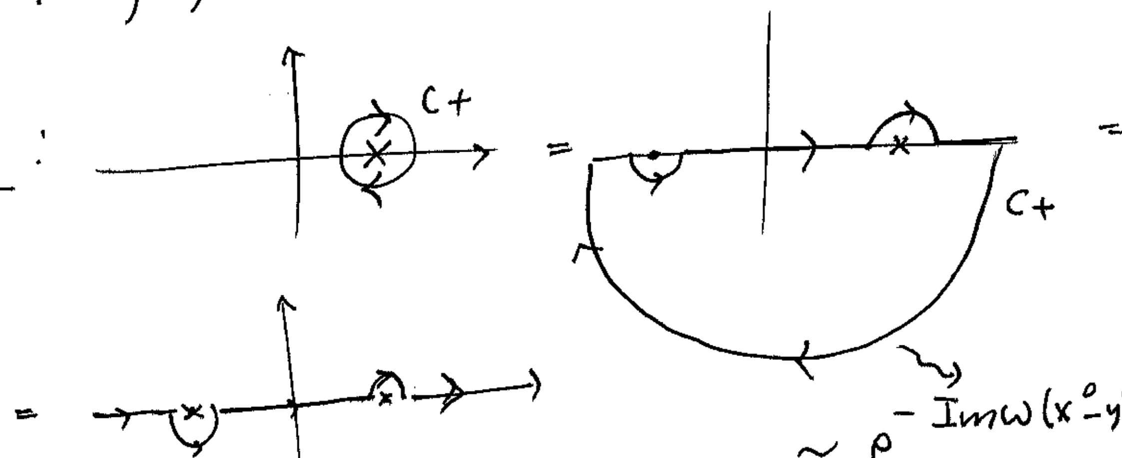
$$\frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu}$$

$$= \theta(x^0-y^0)\int\frac{d^3p}{(2\pi)^3}\int_{C_+}\frac{d\omega}{2\pi}\left(\not{p}_{\alpha\beta}+m\delta_{\alpha\beta}\right)\frac{ie^{-ip(x-y)}}{p^2-m^2}$$

$$+\ \theta(y^0-x^0)\int\frac{d^3p}{(2\pi)^3}\int_{C_-}\frac{d\omega}{2\pi}\left(\not{p}_{\alpha\beta}+m\delta_{\alpha\beta}\right)\frac{ie^{-ip(x-y)}}{p^2-m^2}$$

the integrand is the same. we have to take C_+ for positive x^0-y^0 and C_- for negative x^0-y^0, i.e.

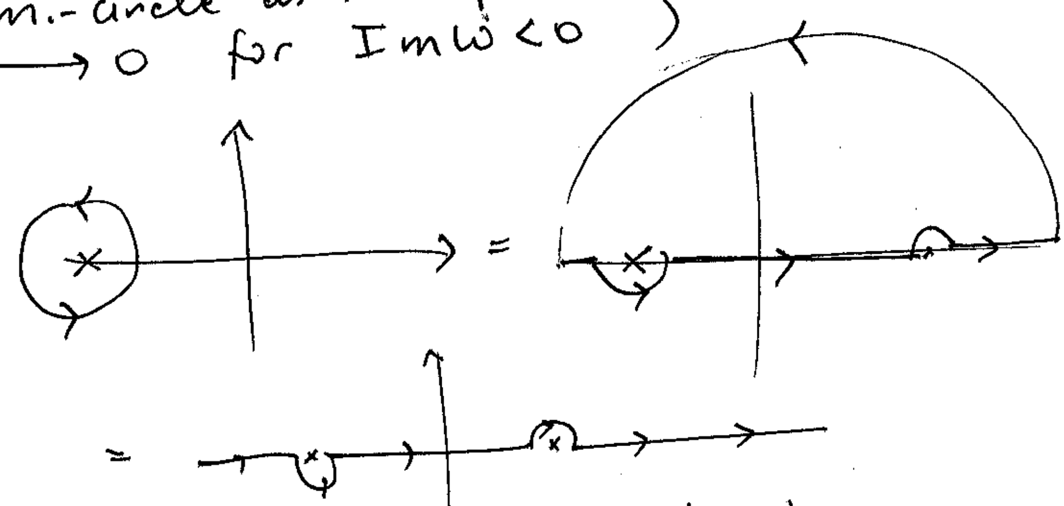
$x^0-y^0>0$:



$\sim e^{-\mathrm{Im}\,\omega(x^0-y^0)} \longrightarrow 0$

(we can ignore the large semi-circle as the exponential $\longrightarrow 0$ for $\mathrm{Im}\,\omega<0$)

$x^0-y^0<0$:



(Again we can ignore the large semi-circle for $\mathrm{Im}\,\omega>0$)

In both cases we have the same contour C_F

$= 0$

$$\langle 0 | T \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = i \int \frac{d^3 p}{(2\pi)^3} \int_{C_F} \frac{d\omega}{2\pi} \frac{(\not{p}_{\alpha\beta} + m\delta_{\alpha\beta})}{p^2 - m^2} e^{-ip(x-y)}$$

$$\equiv S^F_{\alpha\beta}(x-y)$$

4)

$$\left(i \gamma^\mu_{\alpha\beta} \frac{\partial}{\partial x^\mu} - m \delta_{\alpha\beta} \right) \left(-i \, S^F_{(x-y)} \right)_{\beta\gamma} =$$

$$= \left(i \gamma^\mu_{\alpha\beta} \frac{\partial}{\partial x^\mu} - m \delta_{\alpha\beta} \right) \int_{C_F \times \mathbb{R}^3} \frac{d^4 p}{(2\pi)^4} \frac{(\not{p}_{\beta\gamma} + m \delta_{\beta\gamma})}{p^2 - m^2} e^{-ip(x-y)}$$

$$= \int_{C_F \times \mathbb{R}^3} \frac{d^4 p}{(2\pi)^4} \frac{(\not{p}_{\alpha\beta} - m\delta_{\alpha\beta})(\not{p}_{\beta\gamma} + m\delta_{\beta\gamma})}{p^2 - m^2} e^{-ip(x-y)}$$

Use $(\not{p} - m)_{\alpha\beta} (\not{p} + m)_{\beta\gamma} = (p^2 - m^2) \mathbb{1}_{\alpha\gamma}$

$$= \int \frac{d^4 p}{(2\pi)^4} \delta_{\alpha\gamma} \, e^{-ip(x-y)}$$

$$= \delta_{\alpha\gamma} \, \delta^{(4)}(x-y)$$