QFT - 2023–24 Problem Set 3

1] $$\mathcal{L} = \sum_{i=1}^{3} (\partial_\mu \phi_i)^* (\partial^\mu \phi_i) - V(\phi_i)$$

$$V = m^2 \left(|\phi_1|^2 + |\phi_2|^2 \right) + \frac{\lambda}{3} \left(|\phi_3|^2 - \mu^2 \right)^2$$
$$+ v \left(\phi_1^* \phi_2 \phi_3^2 + \phi_1 \phi_2^* \phi_3^{*2} \right)$$

1) - the kinetic term is obviously invariant
- the m^2- and λ- terms of the potential are obviously invariant
- the v- terms transform as:

$$\phi_1^* \phi_2 \phi_3^2 \longrightarrow (e^{-i\theta_1} \phi_1^*)(e^{i\theta_2} \phi_2)(e^{i\frac{\theta_1-\theta_2}{2}} \phi_3)^2$$
$$= \cancel{e^{-i\theta_1}}\, \cancel{e^{i\theta_2}}\, \cancel{e^{i(\theta_1-\theta_2)}}\, \phi_1^* \phi_2 \phi_3$$

same for the complex conjugate term

$\Rightarrow$ $\mathcal{L}$ is invariant.

If $v=0$ $\mathcal{L}$ is invariant under 3 independent phase rotations: $\phi_i \to e^{i\theta_i} \phi_i \quad i=1,2,3$

$v \neq 0$ breaks explicitly this symmetry:

$$U(1) \times U(1) \times U(1) \longrightarrow U(1)_1 \times U(1)_2$$

2. In general: $\epsilon J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi$

We have 2 transformations:

(For infinitesimal θ : $e^{i\theta} \simeq 1 + i\theta$)

1: $\delta_1 \phi_1 = i\theta_1 \phi_1$, $\delta_1 \phi_2 = 0$, $\delta_1 \phi_3 = i\frac{\theta_1}{2} \phi_3$

2: $\delta_2 \phi_1 = 0$, $\delta_2 \phi_2 = i\theta_2 \phi_2$, $\delta_2 \phi_3 = -\frac{i\theta_2}{2} \phi_3$

$$\theta_1 J_1^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_1} \delta \phi_1 + \frac{\delta \mathcal{L}}{\partial \partial_\mu \phi_2} \underbrace{\delta \phi_2}_{=0} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_3} \delta \phi_3$$

$$+ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_1^*} \delta \phi_1^* + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_2^*} \underbrace{\delta \phi_2^*}_{=0} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_3^*} \delta \phi_3^*$$

$$= (\partial^\mu \phi_1)^* (i\theta_1 \phi_1) + (\partial^\mu \phi_3)^* (\frac{i\theta_1}{2} \phi_3)$$

$$+ (\partial^\mu \phi)(-i\theta_1 \phi_1^*) + (\partial^\mu \phi_3)(-\frac{i\theta_1}{2} \phi_3^*) =$$

$$= \theta_1 [-i(\phi_1^* \partial^\mu \phi - \phi_1 \partial^\mu \phi_1^*) - \frac{i}{2}(\phi_3^* \partial^\mu \phi_3 - \phi_3 \partial^\mu \phi_3^*)]$$

$$\Rightarrow \boxed{J_1^\mu = -i(\phi_1^* \partial^\mu \phi - \phi_1 \partial^\mu \phi_1^*) - \frac{i}{2}(\phi_3^* \partial^\mu \phi_3 - \phi_3 \partial^\mu \phi_3^*)}$$

similarly:

$$\boxed{J_2^\mu = -i(\phi_2^* \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_2^*) + \frac{i}{2}(\phi_3^* \partial^\mu \phi_3 - \phi_3 \partial^\mu \phi_3^*)}$$

3) We want to solve :

$$\frac{\partial V}{\partial \phi_1} = 0 \qquad \frac{\partial V}{\partial \phi_2} = 0 \qquad \frac{\partial V}{\partial \phi_3} = 0$$

$$\frac{\partial V}{\partial \phi_1^*} = 0 \qquad \frac{\partial V}{\partial \phi_2^*} = 0 \qquad \frac{\partial V}{\partial \phi_3^*} = 0$$

Since V is real, we can pick 3 of these equations:

$$\frac{\partial V}{\partial \phi_1^*} = 0 = m^2 \phi_1 + \sigma \phi_2 \phi_3^2 \qquad (1)$$

$$\frac{\partial V}{\partial \phi_2^*} = 0 = m^2 \phi_2 + \sigma \phi_1 \phi_3^{*2} \qquad (2)$$

$$\frac{\partial V}{\partial \phi_3} = 0 = \frac{\lambda}{2}\left(\phi_3^* \phi_3 - \mu^2\right)\phi_3^* + 2\sigma \phi_1^* \phi_2 \phi_3 \qquad (3)$$

From eq. (1) we can extract $\phi_1 = -\frac{V}{m^2} \phi_2 \phi_3^2$

$\Rightarrow$ eq (2) becomes : $m^2 \phi_2 - \frac{V^2}{m^2} \phi_2 (\phi_3^* \phi_3)^2 = 0$

this eq. has 2 classes of solution:

a) $\phi_2 = 0$ which also implies $\phi_1 = 0$. ϕ_3 is arbitrary.

b) $\phi_2 \neq 0 \Rightarrow |\phi_3|^2 = \frac{m^2}{V} \Rightarrow \phi_3 = \frac{m}{\sqrt{V}} e^{i\alpha}$

α arbitrary

and then $\phi_1 = -\phi_2 e^{2i\alpha}$

Case a) : eq. (3) becomes :

$$\frac{\lambda}{2}\left(\phi_3^*\phi_3-\mu^2\right)\phi_3^* = 0 \Rightarrow \begin{cases} \phi_3 = 0 \\ \text{or} \\ |\phi_3| = \mu \Rightarrow \phi_3 = \mu e^{i\beta} \end{cases}$$

Case b) : eq. (3) becomes :

$$\frac{\lambda}{2}\left(|\phi_3|^2-\mu^2\right)\phi_3^* - 2\frac{V^2}{m^2}\left(\phi_2\phi_3^2\right)^*\phi_2\phi_3 = 0$$

Replace $\phi_3 = \frac{m}{\sqrt{V}} e^{i\alpha}$

$$\Rightarrow \frac{\lambda}{2}\left(\frac{m^2}{V}-\mu^2\right)\frac{m}{\sqrt{V}}\cancel{e^{-i\alpha}} - 2m\sqrt{V}\,|\phi_2|^2\cancel{e^{-i\alpha}} = 0$$

$$\Rightarrow |\phi_2|^2 = \frac{\lambda}{2V}\left(\frac{m^2}{V}-\mu^2\right)$$

let us call this ξ^2

We must require $\frac{m^2}{V} > \mu^2$ for this to exist.

$$\Rightarrow \phi_2 = \xi e^{i\gamma}.$$

$$\phi_1 = -\xi e^{i\gamma+2i\alpha}$$

and : $$\phi_3 = \frac{m}{\sqrt{V}} e^{i\alpha}$$

To summarize

there are 3 types of solutions:

a1): $\boxed{\phi_1 = \phi_2 = \phi_3 = 0}$

a2): $\boxed{\phi_1 = \phi_2 = 0 \,; \quad \phi_3 = \mu e^{i\beta}}$ $\quad \beta$ arbitrary phase

b) (only exists if $\frac{m^2}{v} > \mu^2$)

$$\boxed{\begin{cases} \phi_1 = -\xi e^{i\gamma + 2i\alpha} \\ \phi_2 = \xi e^{i\gamma} \\ \phi_3 = \frac{m}{\sqrt{v}} e^{i\alpha} \end{cases}}$$

$$\xi = \sqrt{\frac{\lambda}{2v}\left(\frac{m^2}{v} - \mu^2\right)}$$

α, γ arbitrary phases.

4) consider the three types of solution one by one

a1): $\phi_1 = \phi_2 = \phi_3 = 0$ is invariant under $U(1)_1 \times U(1)_2$ $\Rightarrow$ no symmetry breaking

$\Rightarrow$ no Goldstone Bosons

the quadratic Lagrangian is:

$$\mathcal{L} = \sum \partial_\mu \phi_i^* \partial^\mu \phi_i \underbrace{- m^2 |\phi_1^{(x)}|^2 - m^2 |\phi_2^{(x)}|^2}_{= 4 \text{ massive real scalar with SAME mass}} + \underbrace{\frac{\lambda}{2}\mu^2}_{\substack{\text{wrong sign mass term} \\ \Rightarrow 2 \text{ real tachyons}}} |\phi_3^{(x)}|^2 - \frac{\mu^4 \lambda}{4}$$

a2)

Under a generic transformation with parameters θ_1, θ_2:

$$\begin{matrix} \phi_1 = 0 \\ \phi_2 = 0 \\ \phi_3 = \mu e^{i\alpha} \end{matrix} \longrightarrow \begin{matrix} 0 \\ 0 \\ \mu \underbrace{e^{i\alpha - \frac{i\theta_1}{2} + \frac{i\theta_2}{2}}}_{= e^{i\alpha} \text{ iff } \theta_1 = \theta_2} \end{matrix}$$

$\Rightarrow$ the $U(1)_D$ with $\theta_1 = \theta_2 = \theta$ is preserved:

$$\phi_1 \to e^{i\theta}\phi_1, \quad \phi_2 = e^{i\theta}\phi_2, \quad \phi_3 \to \phi_3$$

$$U(1)_1 \times U(2) \xrightarrow[\text{to}]{\text{Broken}} U(1)_D$$

$\Rightarrow$ 1 symmetry broken $\Rightarrow$ 1 Goldstone Boson

The Goldstone boson is the excitation in the direction of the broken symmetry.

Parametrize fluctuations as:

$$e^{\frac{i\pi(x)}{\mu}}\phi_1(x), \quad e^{\frac{-i\pi(x)}{\mu}}\phi_2(x), \quad \left(\mu + \frac{\rho(x)}{\sqrt{2}}\right) e^{i\alpha + i\frac{\pi(x)}{\mu}}$$

$\pi(x)$ is the Goldstone boson

Check: $\pi(x)$ does not appear in the potential:

$$V(\phi_1, \phi_2, \phi_3) = -m^2|\phi_1(x)|^2 - m_2^2|\phi_2(x)|^2 - \frac{\lambda}{4}\left(\left(\mu + \frac{\rho(x)}{\sqrt{2}}\right)^2 - \mu^2\right)^2$$

$$-\,\nu\left(\phi_1^* \cancel{e^{\frac{-i\pi(x)}{\mu}}}\,\phi_2 \cancel{e^{\frac{-i\pi(x)}{\mu}}}\left(\mu + \frac{\rho}{\sqrt{2}}\right)^2 \cancel{e^{\frac{2i\pi(x)}{\mu}}} + \text{complex conj.}\right)$$

to quadratic order in ρ, ϕ_1, ϕ_2:

$$= -m^2|\phi_1|^2 - m^2|\phi_2|^2 - \frac{\lambda}{2}\mu^2\rho^2 - v\mu^2\left(\underline{\phi_1^*\phi_2 + \phi_2^*\phi_1}\right)$$

$\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbb{R}$

Write $\phi_1 = \frac{\varphi_1 + i\psi_1}{\sqrt{2}}$, $\phi_2 = \frac{\varphi_2 + i\psi_2}{\sqrt{2}}$

$$= -\frac{m^2}{2}(\varphi_1^2 + \psi_1^2 + \varphi_2^2 + \psi_2^2) - \frac{\lambda}{2}\mu^2\rho^2 - 2\mu^2 v\left[\varphi_1\varphi_2 + \psi\psi_2\right]$$

$$= -\frac{1}{2}(\varphi_1 \ \varphi_2)\begin{pmatrix} m^2 & \mu^2 v \\ \mu^2 v & m^2 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} - \frac{1}{2}(\psi_1 \ \psi_2)\begin{pmatrix} m^2 & \mu^2 v \\ \mu^2 v & m^2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{\lambda}{2}\mu^2\rho^2$$

the fields $\varphi_1(x), \varphi_2(x)$ mix. we have to find the eigenvalues M^2 of the mass matrix:

$$\begin{pmatrix} m^2 & \mu^2 v \\ \mu^2 v & m^2 \end{pmatrix} : \quad \det\begin{pmatrix} m^2 - M^2 & \mu^2 v \\ \mu^2 v & m^2 - M^2 \end{pmatrix} = 0$$

$$\Rightarrow (m^2 - M^2)^2 - \mu^2 v^2 = 0 \Rightarrow M^2 = m^2 \pm \mu^2 v$$

so the spectrum consists of:

- 2 Real massive scalars with mass $m^2 + \mu^2 v$
- 2 Real massive scalars with mass $m^2 - \mu^2 v$
- 1 Real massive scalar $\rho(x)$ with mass $\lambda\mu^2$
- 1 Real <u>massless</u> scalar $\pi(x)$ (Goldstone Boson)

If $m^2 > \mu^2 |v|$ none of these is a Tachyon and the ground states of type a2 are <u>minima</u>

b) Choose for simplicity the vacuum with $\alpha = \gamma = 0$

(it does not make any difference, you can check).

$$\phi_1^0 = -\xi \,, \quad \phi_2^0 = \xi \,, \quad \phi_3^0 = \frac{m}{\sqrt{v}} \qquad \xi \equiv \sqrt{\frac{\lambda}{2v}\left(\frac{m^2}{v} - \mu^2\right)}$$

the Goldstone bosons are the excitation in the direction of the broken symmetry

(replace $\theta_1 \to \pi_1(x)$, $\theta_2 \to \pi_2(x)$ in the symmetry transformation)

$$\phi_1(x) = \left(-\xi + \frac{\varphi_1(x)}{\sqrt{2}}\right) e^{\frac{i\pi_1(x)}{\xi}}$$

$$\phi_2(x) = \left(\xi + \frac{\varphi_2(x)}{\sqrt{2}}\right) e^{\frac{i\pi_2(x)}{\xi}}$$

$$\phi_3(x) = \left(\frac{m}{\sqrt{v}} + \frac{\varphi_3(x)}{\sqrt{2}} + i\frac{\psi_3(x)}{\sqrt{2}}\right) e^{-i\frac{\pi_1(x)}{2\xi} + \frac{\pi_2(x)}{2\xi}}$$

with $\varphi_1, \varphi_2, \varphi_3, \psi_3, \pi_1, \pi_2$ real $=$ 6 degrees of freedom

the potential is:

$$V(\phi_1, \phi_2, \phi_3) = -m^2\left(\frac{\varphi_1}{\sqrt{2}} - \xi\right)^2 - m^2\left(\frac{\varphi_2}{\sqrt{2}} + \xi\right)^2 -$$

$$- \frac{\lambda}{4}\left(\left(\frac{m}{\sqrt{v}} + \frac{\varphi_3}{\sqrt{2}}\right)^2 + \frac{\psi_3^2}{2} - \mu^2\right)^2 +$$

$$- 2v\,\mathrm{Re}\left[\left(\frac{\varphi_1}{\sqrt{2}} - \xi\right)\left(\frac{\varphi_2}{\sqrt{2}} + \xi\right)\left(\frac{\varphi_3 + i\psi_3}{\sqrt{2}} + \frac{m}{\sqrt{v}}\right)^2\right]$$

Expand to quadratic order in $\varphi_1, \varphi_2, \varphi_3, \psi_3$

$$V = -2m^2\xi^2 + \frac{2m^2}{\sqrt{2}}\varphi_1\xi - \frac{2m^2}{\sqrt{2}}\varphi_2\xi - \frac{m^2}{2}\varphi_1^2 - \frac{m^2}{2}\varphi_2^2$$

$$-\frac{\lambda}{4}\left[\frac{m^2}{v} - \mu^2 + 2\frac{m}{\sqrt{2v}}\varphi_3 + \frac{\varphi_3^2}{2} + \frac{\psi_3^2}{2}\right]^2 +$$

$$-2v\,\mathrm{Re}\left[\left(-\xi^2 + \frac{1}{\sqrt{2}}(\varphi_1-\varphi_2)\xi + \frac{\varphi_1\varphi_2}{2}\right)\left(\frac{m^2}{v} + \frac{2m}{\sqrt{2v}}\varphi_3 + i\psi_3 + \frac{1}{2}\varphi_3^2 - \frac{1}{2}\psi_3^2 + 2i\varphi_3\psi_3\right)\right]$$

$$= \cancel{-2m^2\xi^2} - \frac{\lambda}{4}\left(\frac{m^2}{v} - \mu^2\right)^2 + \cancel{2m^2\xi^2}$$

zeroth order term

$$\left.\begin{aligned} &+\cancel{\frac{2m^2}{\sqrt{2}}\xi(\varphi_1-\varphi_2)} - \cancel{\lambda\left(\frac{m^2}{v}-\mu^2\right)\frac{m}{\sqrt{2v}}\varphi_3} \\ &-\cancel{\frac{2m^2}{\sqrt{2}}\xi(\varphi_1-\varphi_2)} + \cancel{2\xi^2 v\frac{m}{\sqrt{2v}}\varphi_3}\end{aligned}\right] = 0$$

linear terms

$$\xi = \frac{\lambda}{2v}\left(\frac{m^2}{v} - \mu^2\right)$$

$$-\frac{m^2}{2}\varphi_1^2 - \frac{m^2}{2}\varphi_2^2 - \frac{\lambda}{4}\left(\frac{m^2}{v} - \mu^2\right)(\varphi_3^2 + \psi_3^2) - \frac{\lambda}{2}m^2\varphi_3^2$$

$$- m^2\varphi_1\varphi_2 - \sqrt{2}\,v\xi(\varphi_1-\varphi_2)\frac{\sqrt{2}\,m}{\sqrt{v}}\varphi_3 + \xi^2 v(\varphi_3^2 - \psi_3^2)$$

use $\xi = \sqrt{\frac{\lambda}{2v}\left(\frac{m^2}{v} - \mu^2\right)}$

Quadratic term $$= -\frac{3}{4}\lambda\left(\frac{m^2}{v} - \mu^2\right)\psi_3^2 + \frac{1}{4}\lambda\left(\frac{m^2}{v} - \mu^2 - 2m^2\right)\varphi_3^2$$

$$-\frac{m^2}{2}(\varphi_1+\varphi_2)^2 \qquad - 2m\sqrt{v}\,\xi\,\varphi_3(\varphi_1-\varphi_2)$$

Define: $\varphi_1 + \varphi_2 = \psi_1$, $\varphi_1 - \varphi_2 = \psi_2$

$$= -\frac{m^2}{2}\psi_1^2 - \frac{3}{4}\lambda\left(\frac{m^2}{v} - \mu^2\right)\psi_3^2 + \frac{1}{4}\lambda\left(\frac{m^2}{v} - \mu^2 - 2m^2\right)\varphi_3^2 - 2m\sqrt{v}\xi\varphi_3\psi_2$$

$\Rightarrow$ we have 4 massive real fields.

- ψ_3 with mass $m_3^2 = \frac{3}{2}\lambda\left(\frac{m^2}{v} - \mu^2\right) > 0$
- ψ_1 with mass $m^2 > 0$

the other two masses are the eigenvalues of the matrix that mixes φ_3 and ψ_2:

$$\mathcal{M} = \begin{pmatrix} -\frac{\lambda}{2}\left(\frac{m^2}{v} - \mu^2 - 2m^2\right) & m\xi\sqrt{v} \\ m\xi\sqrt{v} & 0 \end{pmatrix}$$

$$\det(\mathcal{M} - \mathbb{1}M^2) = M^2\left(\frac{\lambda}{2}\left(\frac{m^2}{v} - \mu^2 - 2m^2\right) + M^2\right) - m^2\xi^2 v = 0$$

$$M^4 + \frac{\lambda}{2}\left(\frac{m^2}{v} - \mu^2 - 2m^2\right)M^2 - m^2\xi^2 v = 0$$

$$\rightarrow M^2 = -\frac{\lambda}{4}\left(\frac{m^2}{v} - \mu^2 - 2m^2\right) \pm \sqrt{\frac{\lambda^2}{4}\left(\frac{m^2}{v} - \mu^2 - 2m^2\right)^2 + m^2\xi^2 v}$$

Weather these are both real and > 0 depends on the parameters.

2] 1) The field equation for A_μ is

$$\partial^\mu F_{\mu\nu} = J_\nu$$

$$J_\nu \equiv -\frac{\delta \mathcal{L}(\phi)}{\delta A^\nu}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}(\phi)$$
$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} + \mathcal{L}(\phi)$$
$$= -\frac{1}{2}(\partial_\mu A_\nu)F^{\mu\nu} + \mathcal{L}(\phi)$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial \mathcal{L}(\phi)}{\partial A_\nu} \Rightarrow \partial_\mu F^{\mu\nu} = -\frac{\partial \mathcal{L}(\phi)}{\partial A_\nu} \equiv J_\nu$$

where

$$\mathcal{L}(\phi) = (D_\nu\phi)^\dagger(D^\nu\phi) + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 =$$

$$D\phi \equiv \partial_\nu\phi - ieA_\nu\phi$$

$$= (\partial_\nu\phi)^\dagger(\partial^\nu\phi) + ieA_\nu(\phi^\dagger\partial^\nu\phi - (\partial^\nu\phi^\dagger)\phi) + e^2A_\nu A^\nu\phi^\dagger\phi$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial A^\nu} = ie(\phi^\dagger\partial_\nu\phi - (\partial_\nu\phi^\dagger)\phi) + 2e^2A_\nu\phi^\dagger\phi$$

$$= ie\left(\phi^\dagger(\partial_\nu - ieA_\nu)\phi - [(\partial_\nu + ieA_\nu)\phi^\dagger]\phi\right)$$

Now take the component i of the Field eq:

$$\partial^\mu F_{\mu i} = J_i \qquad \vec{J} = -ie\left(\phi^\dagger(\vec{\nabla} - ie\vec{A})\phi - [(\vec{\nabla} + ie\vec{A})\phi^\dagger]\phi\right)$$

For a static configuration : $\partial^0 F_{0i} = \partial_t E_i = 0$

$$\Rightarrow \partial^j F_{ji} = J_i \qquad \text{with} \qquad F_{ji} = \epsilon_{jik}B_k$$

$$\Rightarrow \underbrace{\epsilon_{jik}}_{-\epsilon_{ijk}}\underbrace{\partial^j}_{(-\partial_j)} B_k = J_i \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \vec{J}}$$

2) if $\mu^2 > 0$ the vacuum breaks the gauge symmetry:

$$\frac{\partial V}{\partial(\phi^\dagger\phi)} = 0 \Rightarrow \frac{\mu^2}{2} - \frac{\lambda}{2}(\phi_0^\dagger\phi_0) = 0 \Rightarrow |\phi_0|^2 = \frac{\mu^2}{\lambda}$$

Choose $\phi_0 = v = \sqrt{\frac{\mu^2}{\lambda}}$

In the vacuum configuration $\vec{\nabla}\phi_0 = 0$ and:

$$\vec{J} = -2e^2\vec{A}\,\phi_0^\dagger\phi_0 = -2e^2v^2\vec{A}$$ London equation

the field equation is: $\vec{\nabla}\times\vec{B} = -2e^2v^2\vec{A}$

take another $\vec{\nabla}\times$:

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times B)}_{-\nabla^2\vec{B}+\vec{\nabla}(\vec{\nabla}\cdot\vec{B}),\ \vec{\nabla}\cdot\vec{B}=0} = -2e^2v^2\underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}} \Rightarrow \boxed{\nabla^2\vec{B} = 2e^2v^2\vec{B}}$$

3) If $\vec{A} \neq 0$ then $\vec{J} \neq 0$ but the electric field is $\vec{E} = \partial_t\vec{A} = 0$

$$\Rightarrow \boxed{\rho = 0}$$

SU(2) Breaking by a doublet scalar

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad \phi \to U\phi \quad \text{or} \quad \delta\phi = i\sum_{a=1}^{3} \alpha^a \frac{\sigma^a}{2}\phi$$

suppose ϕ gets a vacuum expectation value (no matter why) : $\phi^\dagger\phi = |\varphi_1|^2 + |\varphi_2|^2 = v^2$

1) Choose the vacuum $\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$

the infinitesimal transformation acts on this as:

$$\delta\Phi_0 = i\begin{pmatrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = v\begin{pmatrix} \alpha^1 - i\alpha^2 \\ \alpha^3 \end{pmatrix}$$

the only solution to $\delta\Phi_0 = 0$ is $\alpha^1 = \alpha^2 = \alpha^3 = 0$ (remember that the $\alpha^a \in \mathbb{R}$), i.e. no transformation leaves the vacuum invariant

$\Rightarrow$ SU(2) is completely broken

"$SU(2) \longrightarrow \mathbb{1}$"

2) $$D_\mu \phi = \partial_\mu \phi - i g A^a_\mu \tau^a \phi \qquad (\text{“} \textstyle\sum_a \text{” understood})$$

$$= \left(\mathbb{1}_2 \partial_\mu - \frac{i}{2} g \begin{bmatrix} A^3_\mu & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu & -A^3_\mu \end{bmatrix} \right) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

3 vector fields $A^1_\mu, A^2_\mu, A^3_\mu$

3) Setting $\Phi = \Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$:

$$D_\mu \Phi_0 = -\frac{i}{2} g \begin{pmatrix} A^3_\mu & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu & -A^3_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = -\frac{i g v}{2} \begin{pmatrix} A^1_\mu - i A^2_\mu \\ -A^3_\mu \end{pmatrix}$$

the kinetic term, evaluated on $\phi = \Phi_0$, is:

$$(D_\mu \Phi_0)^\dagger (D^\mu \phi) = \frac{g^2 v^2}{4} \left(A^1_\mu + i A^2_\mu, \ -A^3_\mu \right) \begin{pmatrix} A^{1\mu} - i A^{2\mu} \\ -A^{3\mu} \end{pmatrix}$$

$$= \frac{g^2 v^2}{4} \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu} + A^3_\mu A^{3\mu} \right)$$

so all 3 gauge bosons get the same mass,

$$m_A^2 = \frac{1}{2} g^2 v^2$$

SU(2) breaking by a Triplet scalar

Remember that $SU(2) \simeq SO(3)$, and the 3-dim representation of $SU(2)$ is the vectorial representation of $SO(3)$:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \Phi \to R\Phi \quad \text{where } R \text{ is a 3-d Rotation}$$

(ϕ_i Real)

$$\delta\Phi = i\alpha^a T^a \Phi$$

these are ~~not~~ the space coordinates but some "internal" labels

T^a are the generators of infinitesimal rotations around x, y, z (Taken Hermitian, i.e. purely imaginary)

Say Φ Takes a v.e.v., $\Phi^t \Phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = v^2$

1) Choose vacuum $\Phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

Under a generic transformation:

$$\delta\Phi_0 = \alpha^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + \alpha^3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$$= \alpha^1 \begin{pmatrix} 0 \\ -v \\ 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} + \alpha^3 \cdot 0 = v \begin{pmatrix} \alpha^2 \\ -\alpha^1 \\ 0 \end{pmatrix}$$

So $\delta\phi_0 = 0$ requires $\alpha^1 = \alpha^2 = 0$ <u>but</u> α^3 can be $\neq 0$

$\Rightarrow$ Transformations generated by T^3 <u>do not</u> break the symmetry

$$\delta_3 \phi_0 \equiv \alpha^3 T^3 \phi_0 = 0$$

These transformations form an $SO(2)$ subgroup of $SO(3)$: the rotations around the 3 axis:

$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_3(\theta)\phi_0 = \phi_0$$

$$\text{“} SO(3) \longrightarrow SO(2) \text{”}$$

or in terms of the universal covering groups:

$$\text{“} SU(2) \longrightarrow U(1) \text{”}$$

$SU(2)$ is broken down to $U(1)$

To see it in terms of $SU(2)$ and $U(1)$ arrange the triplet Φ into a 2×2 complex matrix:

$$\Phi = \begin{pmatrix} \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & -\phi^3 \end{pmatrix}$$

$SU(2)$ acts by adjoint action:

$$\phi \to U\phi U^{-1} \qquad \delta\phi = i\alpha^a [\tau^a, \phi]$$

(finite) (infinitesimal)

if $\phi_0 = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$ $\quad \delta\phi_0 = 0$ if $\alpha^1 = \alpha^2 = 0$ $\left([\sigma^3, \phi_0] = 0\right)$

The unbroken $U(1)$ acts as: $\phi \to e^{i\theta\frac{\sigma^3}{2}} \phi \, e^{-i\theta\frac{\sigma^3}{2}}$

$$\delta\phi = i\theta \left[\tfrac{\sigma^3}{2}, \phi\right]$$

2) $D_\mu \phi = \partial_\mu \phi - ig A^a_\mu T^a \phi =$

$$= \left[\mathbb{1}_3 \partial_\mu - g \begin{pmatrix} 0 & -A^3_\mu & A^2_\mu \\ A^3_\mu & 0 & -A^1_\mu \\ -A^2_\mu & A^1_\mu & 0 \end{pmatrix} \right] \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

3) set $\phi = \phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

$$D_\mu \phi = -g \begin{pmatrix} 0 & -A^3_\mu & A^2_\mu \\ A^3_\mu & 0 & -A^1_\mu \\ -A^2_\mu & A^1_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = gv \begin{pmatrix} A^2_\mu \\ -A^1_\mu \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}(D_\mu \phi_0)^\dagger (D^\mu \phi_0) = \frac{1}{2} g^2 v^2 \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu} \right)$$

- only A^1_μ and A^2_μ obtain a mass, $m_A = gv$
- A^3_μ stays massless

We learned that, for an SU(2) gauge group:

- a doublet breaks SU(2) completely
- a triplet leaves a U(1) subgroup unbroken
- The vector bosons mass is $\div\ gv$