## Useful formulae

- Dirac delta in one dimension:

$$\int_{-\infty}^{+\infty} dy\, e^{iyx} = 2\pi\delta(x); \qquad \frac{d}{dx}\theta(x) = \delta(x)$$

- Cauchy's theorem:
  Given a function $f(z)$ of the complex variable $z$, analytic except at some simple poles $z_i$, then:

$$\oint dz f(z) = 2\pi i\, n \sum_j Res f_j \qquad Res f_j \equiv \lim_{z\to z_j} (z - z_j) f(z),$$

  where the integral is over a closed contour, the sum is over the poles inside the contour, and $n$ is the number of times the contour is run counter-clockwise.

  In particular, for a function $g(z)$ analytic around a point $w$ in the complex plane:

$$\oint dz \frac{g(z)}{z - w} = 2\pi i\, n\, g(w)$$

  where the integral is over a closed contour which encircles $w$ (and contains no other pole of $g$) and $n$ is the number of times the integration contour goes around $w$ counter-clockwise.

- Spin sum rules:

$$\sum_{s=1}^{2} u_\alpha^s(p)\bar{u}_\beta^s(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v_\alpha^s(p)\bar{v}_\beta^s(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

- Trace identities:

$$Tr(\mathbf{1}) = 4, \quad Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \quad Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).$$

# 1 QFT miscellanea (8pt)

## 1.1 Shift symmetry

Consider the free Lagrangian for a massless real scalar field $\phi$:

$$S_{free} = \frac{1}{2}\int d^4x\, \partial^\mu\phi\partial_\mu\phi \tag{1}$$

1. Show that the action is invariant under the field transformation:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \tag{2}$$

where $\alpha$ is a real constant.

2. Construct the Noether current associated to the symmetry (2) and show that it is conserved (when the field equations are satisfied).

3. Write all the interaction terms with dimension up to 5 (modulo total derivatives) which one can add to the Lagrangian while respecting the symmetry (2).

## 1.2 $\theta$-term in electromagnetism

Consider free electromagnetism (described by the Maxwell Lagrangian) with the addition of the $\theta$-*term*:

$$L_\theta = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \theta\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\nu\sigma}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

where $\theta$ is a real parameter and $\epsilon_{\mu\nu\rho\sigma}$ is the four-dimensional totally antisymmetric symbol, i.e. $\epsilon_{\mu\nu\rho\sigma} = 1$ if $\{\mu\nu\rho\sigma\}$ is an even permutation of $\{0123\}$, $\epsilon_{\mu\nu\rho\sigma} = -1$ if if $\{\mu\nu\rho\sigma\}$ is an odd permutation of $\{0123\}$, and $\epsilon_{\mu\nu\rho\sigma} = 0$ if any two indices are equal.

1. Show that the $\theta$-term is gauge-invariant.

2. Compute the contribution of the $\theta$-term to the field equations for $A_\mu$. Why, although gauge invariant, this term is never included in electromagnetism?

## 1.3 Scalar field with cubic interaction

*In this exercice, you can keep the discussion at the qualitative level, i.e. ignoring all the numerical factors, signs, etc*

Consider the following Lagrangian for a real massive scalar field:

$$S = \int d^4x \left[\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^2\Box\varphi\right] \tag{4}$$

1. What is the dimension (in units of mass) of the coupling constant $g$? Is this coupling relevant, marginal or irrelevant?

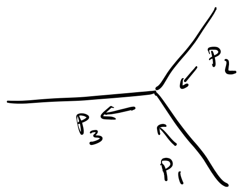
2. The interaction corresponds to a cubic vertex. Compute the action in momentum space by Fourier-transforming:

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4}\tilde{\varphi}(p)e^{-ip\cdot x} \tag{5}$$

and show that the value associate to the vertex is:

3. Write down the Feynman diagram(s) which contribute to lowest order in $g$ to the $\varphi\varphi \to \varphi\varphi$ scattering, and give the corresponding aplitude (keeping the discussion qualitative, i.e. the precise coefficient is not required).

4. Based on the previous result and on dimensional analysis, estimate (qualitatively) the $\varphi\varphi \to \varphi\varphi$ cross section in the high-energy limit as a function of $g$ and the typicall energy $E$ of the process.




$$= g\, p_3^2$$

5. Explain why we cannot use this theory to compute scattering probabilities up to arbitrarily high energy, and estimate the rough energy scale $\Lambda$ at which this theory breaks down.

# 2 Using Green's functions to solve initial value problems (8pt)

In this exercise we want to use the Green's function method to solve the Klein-Gordon initial value problem:

$$\begin{cases} (\Box + m^2)\phi(\vec{x}, t) = 0 \quad \forall t > 0 \\ \phi(\vec{x}, t \to 0^+) = \phi_0(\vec{x}), \\ (\partial_t \phi)(\vec{x}, t \to 0^+) = \phi_1(\vec{x}), \end{cases} . \qquad \Box \equiv \partial_t^2 - \nabla^2 \tag{6}$$

where $\phi_0(\vec{x})$ and $\phi_1(\vec{x})$ are arbitrary functions which satisfy decent boundary conditions (i.e. they decay fast enough) at spatial infinity, $|\vec{x}| \to +\infty$.

Recall that the retarded Green's function $G_R(\vec{x}, t)$ is uniquely defined by the following equations:

$$\left(\Box_{(\vec{x},t)} + m^2\right) G_R(\vec{x}, t) = \delta^{(3)}(\vec{x})\delta(t), \quad G_R(\vec{x}, t < 0) = 0. \tag{7}$$

1. Show that the expression below for $G_R(\vec{x}, t)$ satisfies (7) (the limit $\epsilon \to 0$ is understood):

$$G_R(\vec{x}, t) = -\int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\vec{p}\cdot\vec{x}}}{(\omega + i\epsilon)^2 - \omega_p^2}, \qquad \omega_p \equiv \sqrt{|\vec{p}|^2 + m^2} \tag{8}$$

2. Show that:

$$\lim_{t \to 0^+} G_R(\vec{x}, t) = 0, \qquad \lim_{t \to 0^+} \partial_t G_R(\vec{x}, t) = \delta^{(3)}(\vec{x}) \tag{9}$$

3. Show that the expression

$$\phi(\vec{x}, t) = \int d^3y \left[G_R(\vec{x} - \vec{y}, t)\phi_1(\vec{y}) + \partial_t G_R(\vec{x} - \vec{y}, t)\phi_0(\vec{y})\right] \tag{10}$$

solves the initial value problem (6). Can there be other solutions?

4. Consider now the Klein-Gordon initial value problem with a non-trivial source turned on at time $t = 0$,

$$\begin{cases} (\Box + m^2)\phi(\vec{x}, t) = J(\vec{x}, t) \quad \forall t > 0 \\ \phi(\vec{x}, t \to 0^+) = \phi_0(\vec{x}), \\ (\partial_t \phi)(\vec{x}, t \to 0^+) = \phi_1(\vec{x}), \end{cases} . \tag{11}$$

where we will assume $J(\vec{x}, t \leq 0) = 0$. How does (10) generalize in the presence of this source $J(\vec{x}, t)$?

5. Consider the quantum theory of the Klein-Gordon field. Recall that the retarded Green's function can be written as:

$$G_R(\vec{x}, t) = i\theta(t)\langle 0| \left[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{0}, 0)\right] |0\rangle \tag{12}$$

Show that then properties (9) are a consequence of the identity (12) and of the equal-time canonical commutation relation between $\phi$ and its conjugate momentum.

# 1 Assorted short questions

## 1.1 Dilatation symmetry

Consider the following action for a real scalar field $\phi$:

$$S = \int d^4x \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{g}{4} \phi^4 \right) \tag{1}$$

where $g > 0$ is a constant. Consider the following coordinate transformation (dilatation):

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad \lambda > 0. \tag{2}$$

1. Determine a field transformation law of the form:

$$\phi(x) \to \phi'(x'),$$

   where $\phi'(x')$ is linear in $\phi(x)$ , such that the **action** is invariant under the combined transformation of the field and the dilatation (2).

2. Construct the associated Noether current.

3. Are there any other terms one can add to the action which preserve this symmetry?

## 1.2 SU(2) symmetry

Consider the following Lagrangian for a scalar field doublet $\Phi = \{\phi_i\}$ with $i = 1, 2$ where both $\phi_1$ and $\phi_2$ are complex:

$$L = \frac{1}{2} (\partial^\mu \Phi)^\dagger (\partial_\mu \Phi) - \frac{1}{2} M_1^2 \Phi^\dagger \Phi - \frac{1}{2} M_2^2 \, {}^t\tilde{\Phi} \Phi \tag{3}$$

where $^t$ indicates the transposed and the components of $\tilde{\Phi}$ are $\tilde{\Phi}_i = \epsilon_{ij} \Phi_j$ ($i, j = 1, 2$, sum over repeated indexes is implied, and $\epsilon_{ij}$ is the completely antisymmetric symbol with $\epsilon_{12} = 1$).

1. Show that the Lagrangian (3) is invariant under $SU(2)$ transformations $\Phi \to U\Phi$.

2. Use this result to write down a gauge-invariant (under $SU(2)_L \times U(1)_Y$) interaction between the lepton doublet $L$, the Higgs doublet $H$, and a singlet right-handed neutrino $\nu_R$ in such a way that after symmetry breaking by the Higgs vev the neutrinos acquire a mass (recall that in the standard model the Higgs is an $SU(2)_L$ doublet with $Y$-charge $1/2$, the left-handed lepton doublet ${}^t(\nu_L, e_L)$ has $Y$-charge $-1/2$, and assume the right-handed neutrino singlet has zero $Y$-charge).

### 1.3 Gordon Identity

Let $u(p)$ be a positive frequency solution to the momentum space Dirac Equation,

$$(\gamma^\mu p_\mu - m)u(p) = 0$$

Using this equation, as well as the conjugate equation for $\bar{u}(p)$, prove the following identity:

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')\left[(p'+p)^\mu - \frac{1}{2}[\gamma^\mu, \gamma^\nu](p'-p)_\nu\right]u(p)$$

where $p, p'$ are both on-shell 4-momenta.

## 2 Scalar fields with cubic interaction

*In this exercise, you can keep the discussion at the qualitative level, i.e. ignoring all the numerical factors, signs, etc*

Consider the following Lagrangian for two complex scalar fields:

$$L = \frac{1}{2}(\partial^\mu \varphi)^* \partial_\mu \varphi + \frac{1}{2}(\partial^\mu \chi)^* \partial_\mu \chi - \frac{g}{2}\left(\varphi^2 \chi^* + \varphi^{*2}\chi\right) \tag{4}$$

where $g$ is a real constant.

### 2.1 Symmetry and its breaking

1. Show that the Lagrangian is invariant under the following global symmetry:

$$\varphi \to e^{i\alpha}\varphi, \qquad \chi \to e^{2i\alpha}\chi, \qquad \alpha \in \mathbf{R}. \tag{5}$$

2. What is the group corresponding to this symmetry, and how many infinitesimal generators does it have?

3. Write the infinitesimal transformation(s) corresponding to each symmetry generator

4. Compute the associated Noether current(s).

5. Find the most general classical vacuum (i.e. solution of the classical field equations preserving Poincaré invariance)

6. Are there vacua which preserve the symmetry (5) ? Discuss the particle spectrum (and their masses) around such vacua.

7. Are there vacua which break the symmetry (5) ? Discuss the particle spectrum (and their masses) around such vacua.

## 2.2 Tree-level processes

*We call by the same name the particle and the field and the antiparticle is denoted with a bar. For instance, the particle associated with the field $\varphi(x)$ is called $\varphi$ and its antiparticle $\bar{\varphi}$.*

Assume, for now, that the theory is in the vacuum state in which $\langle\varphi\rangle = \langle\chi\rangle = 0$.

8. Write down the Feynman rules following from the Lagrangian (4).

9. Write down the Feynman diagram(s) (if any) which contribute to lowest order in $g$ to the following processes, and give the corresponding amplitudes (keeping the discussion qualitative, i.e. the precise coefficient is not required). When the amplitude vanishes, give a simple explanation of this fact.

   a) $\varphi\bar{\varphi} \to \chi\bar{\chi}$
   b) $\varphi\varphi \to \varphi\varphi$
   c) $\varphi\varphi \to \varphi\bar{\chi}$

10. Estimate (qualitatively) the $\varphi\varphi \to \varphi\varphi$ cross section as a function of $g$ and the typical energy $E$ of the process ($E \sim |\vec{p}_{in}|$ may be the center of mass energy, or the transferred momentum, or another relevant combination of incoming momenta).

Assume now that the theory is in a symmetry-breaking vacuum, and denote by $v$ the vacuum expectation value of the field which breaks the symmetry.

11. Which of the processes a), b), c) above are possible, and which are not, for incoming energies $E \ll v$?

12. Write the amplitude and, for $E \ll v$, estimate the cross section of each process a), b), c).

# 3 4-Fermion interaction

Consider the following Lagrangian for two kinds of interacting massive spinors $\psi$, $\chi$,

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi + \bar{\chi}(i\gamma^\mu\partial_\mu - m_\chi)\chi - G(\bar{\psi}\psi)(\bar{\chi}\chi) \tag{6}$$

where $G$ is a positive constant.

1. Write down the propagators and the interaction vertices, and give the associated values.

2. What is the dimension of the coupling constant $G$? What is the type of the interaction (relevant, irrelevant, marginal)?

3. Draw the tree-level Feynman diagrams which contribute to the process of pair annihilation/production: $\psi$, anti-$\psi \to \chi$, anti-$\chi$.

4. After specifying each of the external particles momentum and polarization, write the corresponding amplitude *(if you are unsure how to contract the wave-functions, write explicitly the spinor indices on the vertex as they appear in the Lagrangian).*

5. For the unpolarized cross-section, compute the square of the amplitude, summed over final spins and averaged over initial spins.

6. Compute the (unpolarized) scattering cross section in the center-of-mass frame and in the low-energy limit ($E_{CM} \ll m_\psi, m_\chi$). Express the result in terms of the center-of-mass energy.

7. Compute the same quantity in the high-energy limit ($E_{CM} \gg m_\psi, m_\chi$). Can this result be valid for arbitrary $E_{CM}$ ? Up to which energies can we trust the theory (6) ?

# 1 A model for the proton-neutron interaction (18)

In the following, we consider the neutron, the proton and the $\pi$ mesons as if they were elementary particles (therefore ignoring the quarks). This allows us to associate a field with all of them.

We consider a model involving two Dirac fields $\psi_p(x)$ (for the proton) and $\psi_n(x)$ (for the neutron) of the same mass $M$ and a spin 0 real field $\phi(x)$ (for the meson) of mass $m$ which is pseudo-scalar, that is, under the inversion of spatial coordinates it transforms as:

$$x^\mu = (x^0, \vec{x}) \to x'^\mu = (x^0, -\vec{x}), \qquad \phi(x) \to \phi'(x') = -\phi(x).$$

The fields are coupled to each other through the interaction:

$$\mathcal{L}_{\text{int}} = -ig\bar{\psi}_n(x)\gamma^5\psi_n(x)\phi(x) - ig'\bar{\psi}_p(x)\gamma^5\psi_p(x)\phi(x)$$

where $g$ and $g'$ are real constants and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

**The sections of this first problem can be done independently and in no particular order**

## 1.0. Preliminary Question (0.5)

0.a) 0.5 Give a brief argument about the validity of this approach (in which we do as if the neutron, proton and mesons were elementary and we thus associate a quantum field with all of them) and its expected domain of validity.

## 1.1 Lagrangian and Symmetries (4)

1.a) 0.5 Write the full Lagrangian for this theory and identify the free part and the interaction part.

1.b) 0.5 What are the dimensions of $g$ and $g'$?

1.c) 0.5 Show that $\mathcal{L}_{\text{int}}$ is hermitian.

1.d) 1 Recall how parity acts on spinors, and obtain the transformation under parity of the bilinear $\bar{\psi}\gamma^5\psi$.
**Hint:** a possibility is (i) to use the Weyl representation of the Clifford algebra where the Dirac bi-spinors are written $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, (ii) to recall how parity acts on $\psi_L$ and $\psi_R$ (it is realized with one of the $\gamma^\mu$ matrices), and finally (iii) to show that the result does not depend on the

choice of representation of the Clifford algebra.

1.e) 0.5 Is $\mathcal{L}_{\text{int}}$ parity invariant?

1.f) 1 Show that $\mathcal{L}_{\text{int}}$ has two independent $U(1)$ symmetries which act on the proton and the neutron, and write the corresponding Noether currents. What is the interpretation of the corresponding Noether charges, and of their conservation?

## 1.2 Proton-neutron scattering process (8.5)

We consider the scattering process with an initial state with a proton having 4-momentum $p_1$ and spin polarization $s_1$, and a neutron having 4-momentum $p_2$ and spin polarization $s_2$ that evolves into a final state with a proton (4-momentum $p_3$, spin $s_3$) and a neutron (4-momentum $p_4$, spin $s_4$).

2.a) 0.5 Write the Feynman rules for this theory (i.e give the propagators and the vertices).

2.b) 1 Construct the Feynman diagrams contributing to this process.

2.c) 2 Evaluate the scattering amplitude as a function of $p_i$ and $s_i$, i=1,2,3,4.

2.d) 3 Assuming that we do not measure the spin of the final states, and that the initial states are in a random mixture of spin polarizations, give the expression for the differential cross-section in the center-of-mass frame as a function of the Mandelstam invariants $s, t, u$.

2.e) 2 Compute the non-relativistic limit of the cross section as a function of the scattering angle $\theta$.

## 1.3 Isospin symmetry (5)

We want to make the proton-neutron-meson model invariant under a global $SU(2)$ symmetry, under which the proton and neutron form a doublet (the nucleon),

$$\Psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \qquad \Psi_N \to \Psi_N' = U\Psi_N, \quad U \in SU(2).$$

Notice that each component of $\Psi_N$ is a 4-component Dirac spinor, so there is a total of 8 components. However, the matrix $U$ acts as the identity on spinor indices:

$$\Psi_N' = \begin{pmatrix} \psi_p' \\ \psi_n' \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

and, for instance, $(\psi_p')_\alpha = U_{11}(\psi_p)_\alpha + U_{12}(\psi_n)_\alpha$ with $\alpha = 1, \cdots, 4$ a spinor index.
We also assume the meson $\phi = \phi_3$ is part of a triplet of real pseudoscalar fields $(\phi_1, \phi_2, \phi_3)$ arranged in a Hermitian matrix (the meson triplet) transforming under $SU(2)$ as:

$$\Phi = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix}, \qquad \Phi \to \Phi' = U\Phi U^{-1}.$$

We call $T^a$ with $a = 1, 2, 3$ the generators of $SU(2)$ and denote by $I_3$ the eigenvalue of $T^3$. It is called hereafter the isospin.

3.a) 0.5 Write the infinitesimal tranformation of the fields $\Psi_p, \Psi_n$ and $\phi_1, \phi_2, \phi_3$.

3.b) 1 Which fields or which linear combinations of fields have a definite isospin $I_3$ (i.e. are eigenstates of $T^3$?)

3.c) 0.5 Write $SU(2)$-invariant kinetic and mass terms for the multiplets $\Psi_N$ and $\Phi$.

3.d) 1 Write the lowest-dimension $SU(2)$-invariant and parity-invariant interaction term coupling the nucleon doublet and the meson triplet.

3.e) 1 Write the most general parity-invariant and $SU(2)$-invariant potential for the meson, which only contains coupling constants of non-negative mass dimension.

3.f) 1 In the theory above, construct the Feynman diagrams of all possible tree level processes in which a proton and an antiproton can annihilate into two mesons (of any kind) and three mesons (use initial and finite states of definite isospin $I_3$).

# 2 Currents in scalar field theory (12)

We first consider the classical field theory of a complex scalar filed $\phi$. The action is assumed to be invariant under the global $U(1)$ transformation:

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x) \tag{1}$$

with $\alpha$ a real number number. The Lagrangian is

$$L = \partial^\mu \phi^* \partial_\mu \phi - V(\phi^* \phi) \tag{2}$$

and it is $U(1)$-invariant. **Remember that $\phi$ and $\phi^*$ are to be considered as independent quantities.**

## 2.1 Preliminaries (2)

1.a) 0.5 Construct the Noether current $j_\mu$ and show that:

$$j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \tag{3}$$

1.b) 0.5 Define the associated conserved charge $Q$.

1.c) 1 Using the field equations, show explicitly that $\partial^\mu j_\mu = 0$ and explain why this implies $dQ/dt = 0$.

## 2.2 The current and charge in the quantum theory (6)

We now consider the quantum version of the above theory. The field $\phi(x)$ is therefore now a quantum field, that is, an operator. It is not hermitic and the lagrangian of the quantum theory is the same as in (2) up to the replacement of the classical field by its quantum analog and $\phi^*(x)$ by $\phi^\dagger(x)$.

It is quantized by imposing the Equal Time Canonical Commutation Relations (CCR) :

$$[\phi(t,\vec{x}),\Pi_\phi(t,\vec{y})] = [\phi^\dagger(t,\vec{x}),\Pi_{\phi^\dagger}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{4}$$

where $\Pi_\phi$ is the canonical momentum associated with $\phi$ and $\Pi_\phi{}^\dagger$ the canonical momentum associated to $\phi^\dagger$.

The other equal time commutators are vanishing:

$$[\phi(t,\vec{x}),\phi^\dagger(t,\vec{y})] = [\Pi_\phi(t,\vec{x}),\Pi_{\phi^\dagger}(t,\vec{y})] = [\Pi_\phi(t,\vec{x}),\phi^\dagger(t,\vec{y})] = [\Pi_{\phi^\dagger}(t,\vec{x}),\phi(t,\vec{y})] = 0.$$

2.a) 0.5 Obtain from the Lagrangian (2) the expression of $\Pi_\phi$ (in terms of $\phi$ and its derivatives).

2.b) 1 We now want to implement in the quantum case $U(1)$ transformations analogous to the classical transformations in (1). In the first part of the QFT lectures, this was performed for space-time transformations by implementing what was called "the correspondence principle". It amounts to choosing that the average of a quantum operator transforms as its classical analog. In the $U(1)$ case we are interested in, this means:

$$(\langle\Psi|\phi(x)|\Psi\rangle)' = e^{i\alpha}\langle\Psi|\phi(x)|\Psi\rangle \tag{5}$$

and the conjugate relation for $\phi^\dagger(x)$.
As in the lectures, we choose in (5) to transform the states and keep fixed the operators. We assume that a $U(1)$ transofrmation is represented on the Hilbert space by a unitary operator $U(\alpha)$,

$$|\Psi'\rangle = U(\alpha)|\Psi\rangle$$

Deduce from (5) what property $U(\alpha)$ must satisfy.

2.c) 2.5 Using the CCR (4) and the expression for the Noether current, compute the commutator:

$$i\alpha[Q,\phi(x)] \tag{6}$$

where $\alpha$ is a constant and $Q$ is the operator obtained from the classical Noether charge by replacing the classical field by the quantum field. Deduce that $Q$ is the infinitesimal generator of the $U(1)$ symmetry in the quantum case and express the unitary operator $U(\alpha)$ from the previous point in terms of $Q$ (Hint: consider a transformation with $\alpha$ infinitesimal).

2.d) 2 Compute the commutator $[H,Q]$ where $H$ is the hamiltonian and conclude that $Q$ is also conserved in the quantum theory. (Hint: (i) Compute the Hamiltonian, (ii) assume that the potential $V$ in (2) can be expanded in power series, (iii) compute the commutator of a generic term of this series with $\Pi_\phi$ and (iv) deduce that $[\Pi_\phi, V]$ can be written in terms of the derivative of $V$ in a simple way.)

## 2.3 Symmetry breaking (4)

We assume that the theory can be analyzed perturbatively, that is, is close to a free field theory. We denote by $|\Omega\rangle$ the ground state of the theory which is annihilated by all creation operators associated with $\phi$, i.e. $a_{\vec{k}}|\Omega\rangle = 0$. We assume that $|\Omega\rangle$ is an eigenstate of the Hamiltonian with zero energy, $H|\Omega\rangle = 0$.
We suppose now that the $U(1)$ symmetry is spontaneously broken by a vacuum expectation value of the quantum field $\phi$. We want to see what this implies for the quantum theory, and in particular to show that there are massless particles in the specturm (Goldstone Theorem).

The statement of symmetry breaking can be rephrased as:

$$U(1) \text{ spontaneously broken} \qquad \Leftrightarrow \qquad Q|\Omega\rangle \neq 0. \tag{7}$$

3.a) 1 Based on question 2.b) and 2.c), justify the statement (7).

3.b) 1 By taking commutators of $Q$ and $H$ show that if the symmetry is spontantously broken, then the state $Q|\Omega\rangle$ is also a zero-energy eigenstate of $H$.

3.c) 1 We define the state $|\pi(\vec{k})\rangle$:

$$|\pi(\vec{k})\rangle = \int d^3x e^{-i\vec{k}\vec{x}} j_0(t=0,\vec{x})|\Omega\rangle.$$

Using the fact that the momentum operator generates space translations, that is, for any operator $O(x)$,

$$i[P_i, O(x)] = \partial_i O(x), \qquad i = 1,2,3\,,$$

show that $|\pi(\vec{k})\rangle$ is a state with definite 3-momentum $\vec{k}$.

3.d) 1 Using (3.b) show that the energy $\omega_{\vec{k}}$ of the state $|\pi(\vec{k})\rangle$ vanishes when $\vec{k} \to 0$. Deduce that $|\pi(\vec{k})\rangle$ satisfies a massless dispersion relation (hence proving Goldstone theorem at the level of the Hilbert space).

# 1 Fermion propagator

The goal is to derive the expression of the Feynman propagator $S^F_{\alpha\beta}(x,y)$ for a Dirac fermion field of mass $m$, defined as a time-ordered two-point correlation function:

$$S^F_{\alpha\beta}(x,y) = \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle \tag{1}$$

where the time-ordered product of two fermionic operators $A(x)$ and $B(y)$ is defined as follows:

$$TA(x)B(y) = \begin{cases} A(x)B(y) & x^0 > y^0 \\ -B(y)A(x) & x^0 < y^0 \end{cases}$$

You may use the plane wave expansions for free Dirac fields:

$$\psi_\alpha(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left[ b^s_{\vec{p}} u^s_\alpha(p) e^{-i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})} + c^{s\dagger}_{\vec{p}} v^s_\alpha(p) e^{i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right] \tag{2}$$

where $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$, $s$ runs over two polarizations, and $u(p)$ and $v(p)$ are positive and negative frequency spinors satisfying

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

The anti-commutation relations are:

$$\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}), \quad \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}) \tag{3}$$

and all other anti-commutators vanishing.

1. Show that the (unordered) fermion two-point functions can be written as

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} + m\delta_{\alpha\beta}\right) D(x-y), \; \langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle = \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu} - m\delta_{\alpha\beta}\right) D(y-x)$$

where

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} e^{-i(\omega_{\vec{p}}(x^0-y^0) - \vec{p}\cdot(\vec{x}-\vec{y}))}$$

2. Show that, by Cauchy's theorem, $D(x-y)$ can be written as a four-dimensional integral over momentum-space,

$$D(x-y) = \int_{C_+} \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{i}{p^2 - m^2} e^{-ip_\mu(x-y)^\mu} \tag{4}$$

where $p^\mu = (\omega, \vec{p})$ (with $\omega$ independent of $\vec{p}$), $p^2 \equiv p^\mu p_\mu$, and the integration over frequency is performed around an appropriate contour in the complex plane encircling the positive frequency pole on the real axis, $\omega_{\vec{p}} = +\sqrt{|\vec{p}|^2 + m^2}$.

What is the function one obtains by integrating instead around the contour $C_-$ encircling the negative frequency pole, $\omega_{\vec{p}} = -\sqrt{|\vec{p}|^2 + m^2}$ ?

3. Use the results of the previous two points to show that the fermion time-ordered two-point correlator $S^F_{\alpha\beta}(x,y)$, defined in equation (1), is given by the expression

$$S^F_{\alpha\beta}(x,y) = i\int_{C_F}\frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\frac{\not{p}_{\alpha\beta}+m\,\delta_{\alpha\beta}}{p^2-m^2}e^{-ip_\mu(x-y)^\mu}$$

the integral over frequency running along the an appropriate contour $C_F$ which you will specify.

4. Show that $-iS^F_{\alpha\beta}(x-y)$ is a Green's function for the Dirac equation,

$$\Big(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} - m\delta_{\alpha\beta}\Big)\Big(-iS^F_{\beta\gamma}(x,y)\Big) = \delta_{\alpha\gamma}\delta^{(4)}(x-y)$$

# 2 Scalar QED

Consider the theory of a massive complex scalar coupled to a $U(1)$ gauge field,

$$\mathcal{L} = (D^\mu\varphi)^*(D_\mu\varphi) - m^2\varphi^*\varphi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad m^2>0. \tag{5}$$

We denote the gauge coupling ("electric charge") by $e$.

## 2.1 Preliminaries

1. What is the spectrum (i.e. the particle content) of the theory (spin, mass, charges)?
2. Show that the Langrangian is invariant under global $U(1)$ transformations,

$$\varphi \to e^{ie\alpha}\varphi, \qquad \alpha = const.$$

   and give the expression of the corresponding Noether current.

3. Show that with the appropriate definition of the covariant derivative $D_\mu$ and of the transormation of the gauge field $A_\mu$, the Lagrangian is also invariant under *local* $U(1)$ transformations (same as above, but with $\alpha = \alpha(x)$).
4. Separate equation (5) into the sum of a free Lagrangian plus interaction terms
5. What are the dimensions of the interaction terms? Which is the dimension of the coupling constant $e$?
6. Write down the (momentum space) Feynman rules of the theory:
   i. Write the propagators for the vector and scalar (use Feynman Gauge for the vector propagator)
   ii. draw the vertexes corresponding to each interaction, and their associated values (in momentum space).

## 2.2 Photon pair-production

We are interested in the process consisting of a scalar particle-antiparticle annihilation into two photons,

$$\varphi^+\varphi^- \to \gamma\gamma$$

7. Draw the tree-level Feynman diagrams contributing to this process (specifying the external momenta)

8. Write the contribution to the amplitude from each diagram, then give the total amplitude.

9. Give the expression for the center-of-mass cross section as a function of the amplitude squared, the center-of-mass energy, and the momenta of the initial and final particles.

10. From the structure of the amplitude, and from dimensional arguments, estimate the cross section in the center-of-mass frame for ultra-relativistic incoming particles ($E_{CM} \gg m$) (do *not* calculate it)

## 2.3 Symmetry breaking

Consider now the same Lagrangian (5) but with the potential $m^2\varphi^*\varphi$ replaced by

$$V(\varphi, \varphi^*) = \frac{\lambda}{2}\left(v^2 - \varphi^*\varphi\right)^2$$

where $v$ is real and $\lambda > 0$.

11. Find the classical vacua (i.e. the $x^\mu$-indepedent field configurations $\varphi$ satisfying the classical field equations). Which of these vacua are stable?

12. Show that the stable vacua break $U(1)$ gauge symmetry spontanteously.

13. Find the mass of the vector bosons.

# 1 Chiral symmetries and their breaking

Consider a theory with the following field contents: 1 complex scalar $\phi$; 1 left-handed spinor $\psi_L$; 1 right-handed spinor $\chi_R$; with the Lagrangian given by:

$$\begin{aligned} \mathcal{L} \quad &= \; (\partial_\mu \phi)^* (\partial^\mu \phi) \; + \; \epsilon \, \mu^2 |\phi|^2 \; - \; \frac{\lambda}{2} |\phi|^4 \\ &\quad + i \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \; + \; i \bar{\chi}_R \gamma^\mu \partial_\mu \chi_R \; + \; g \left( \phi \, \bar{\psi}_L \chi_R \; + \; \phi^* \, \bar{\chi}_R \psi_L \right), \end{aligned} \tag{1}$$

where $\epsilon = \pm 1$, $\mu$ and $g$ are real constants and $\lambda > 0$. We define two independent $U(1)$ transformations, (which we will call $U(1)_L$ and $U(1)_R$):

$$U(1)_L : \begin{cases} \psi_L \to e^{i\alpha} \psi_L \\ \chi_R \to \chi_R \\ \phi \to e^{i\alpha} \phi \end{cases}, \quad U(1)_R : \begin{cases} \psi_L \to \psi_L \\ \chi_R \to e^{i\beta} \chi_R \\ \phi \to e^{-i\beta} \phi \end{cases} \tag{2}$$

where $\alpha$ and $\beta$ are real parameters.

## 1.1 Global case

1. Show that the $U(1)_L \times U(1)_R$ transformations (2) are a symmetry of the Lagrangian (1), and find the corresponding Noether currents.

2. Rewrite the Lagrangian, the action of the $U(1)_L \times U(1)_R$ symmetry, and the Noether currents in terms of a suitable Dirac spinor $\Psi$ constructed out of $\psi_L$ and $\chi_R$.

3. Find the classical vacua of the theory and discuss their stability, in the two cases $\epsilon = 1$ and $\epsilon = -1$. *(Remark: in general, a spinor has to be zero in the classical vacuum if we want the latter to be Poincaré-invariant).*

4. In each of the above cases, discuss whether the $U(1)_L \times U(1)_R$ symmetry is spontaneously broken, and to which residual subgroup, in the stable vacuum.

5. For both values of $\epsilon$, find the spectrum of excitations (by giving their mass and spin) around the stable vacuum.

## 1.2 Local case

Suppose now we want to make the $U(1)_L \times U(1)_R$ symmetry local. We will assume that the two corresponding gauge couplings are equal.

6. Introduce the appropriate gauge fields, specify their gauge transformations, and write the gauge-invariant generalisation of the Lagrangian (1)

7. For $\epsilon = \pm 1$, discuss the breaking of the gauge symmetry and give the new excitation spectrum (mass, spin and charges) around the stable vacuum.

# 2 Massive vector field

The dynamics of a massive spin-one field is described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

## 2.1 Field equations

1. Show that the action (3) is not invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

2. Write down the Euler-Lagrange equations

3. Show that, for $m \neq 0$, the Euler-Lagrange equations imply that $A_\mu$ must satisfy

$$\partial^\mu A_\mu = 0. \tag{4}$$

   Why is this *not* the same as a gauge-condition?

4. Using the EL equations, show that $A_0$ is not a dynamical degree of freedom but it can be eliminated at each instant of time (i.e. by solving an equation with no time-derivatives on $A_0$) in terms of the spatial components $A_i$.

5. Show that the spatial components $A_i$ each satisfy Klein-Gordon's equation with mass $m$.

## 2.2 Propagator

We now want to write down the propagator for the massive spin-1 theory.

6. Show that, up to total derivatives, the Lagrangian (3) can be rewritten as

$$\mathcal{L} = \frac{1}{2}A^\mu \left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu\right) A^\nu + \frac{1}{2}m^2 A_\mu A^\mu \tag{5}$$

7. We now add a source term to the Lagrangian, of the from

$$L_{source} = -J_\mu A^\mu.$$

   Write the field equation in the presence of the source, and the corresponding equation for the associated Green's function $G_{\mu\nu}(x, y)$.

8. By going to momentum-space, find the propagator $G_{\mu\nu}(p)$ *(Hint: by Lorentz-invariance, $G_{\mu\nu}(p)$ can only be the sum of two types of terms: $G_{\mu\nu}(p) = A(p^2, m)g_{\mu\nu} + B(p^2, m)p_\mu p_\nu$ where $A$ and $B$ are functions of $p^2$ and $m$ to be determined.)*

9. Show that, on-shell (i.e. when $p^2 = m^2$), the propagator satisfies $p^\mu G_{\mu\nu}(p) = 0$, in accordance with the constraint equation (4).

## 2.3 Recovering Gauge invariance

We can make the theory of a massive spin-1 gauge-invariant by introducing an appropriate auxiliary scalar field $\pi(x)$, such that under a gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \qquad \pi(x) \to \pi(x) + \alpha(x).$$

10. Find a gauge-invariant Lagrangian for the fields $A_\mu(x)$ and $\pi(x)$ which is equivalent to the original Lagrangian (3).

11. Discuss the differences and similarities between this procedure and the Higgs mechanism for giving mass to $A_\mu$.