M2 NPAC 2023/2024

QFT - part 2 **Problem set n˚1**

1 Symmetries of the Dirac Lagrangian

Consider the Dirac Lagrangian :

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi \tag{1}$$

1.1 Vector Symmetry

Show that $\mathcal{L}$ is invariant under the internal, rigid $U(1)_V$ transformation :

$$\Psi'(x) = e^{i\theta}\Psi(x), \tag{2}$$

and give the expression of the associated Noether current J^μ_V.

1.2 Axial Transformation

Consider now the transformation $U(1)_A$:

$$\Psi'(x) = e^{i\xi\gamma_5}\Psi(x), \qquad \xi \in \mathbf{R} \tag{3}$$

1. Show that this transformation is a symmetry of the Lagrangian if and only if $m = 0$, and construct the associated Noether current J^μ_A.
2. Go back to the demonstration of Noether's theorem for a transformation which *does not leave the Lagrangian invariant*, and show that in this case we have :

$$\partial_\mu J^\mu = \frac{\delta\mathcal{L}}{\delta\xi}$$

   where $\delta\xi$ is the infinitesimal parameter of the transformation and J^μ is the usual definition of the (would-be) Noether current (i.e. the same expression that one has in the case the transformation is a symmetry).
3. For $m \neq 0$, using the equation of motion find the *non*-conservation equation of the axial current $\partial_\mu J^\mu_A$ and show that, indeed, $\xi\partial_\mu J^\mu_A = \delta\mathcal{L}$,

1.3 Symmetries and L and R spinors

Recall the Weyl decomposition of a Dirac spinor Ψ. In the Weyl basis (in which γ_5 is diagonal),

$$\Psi = \begin{pmatrix} \varphi_l \\ \chi_r \end{pmatrix}. \tag{4}$$

We define the left and right components of Ψ by :

$$\Psi_L = \begin{pmatrix} \varphi_l \\ 0 \end{pmatrix}, \qquad \Psi_R = \begin{pmatrix} 0 \\ \chi_r \end{pmatrix}. \tag{5}$$

This decomposition can be made basis-independent by introducing the left and right projectors :

$$P_L = \frac{1-\gamma_5}{2}, \qquad P_R = \frac{1+\gamma_5}{2} \tag{6}$$

and writing :

$$\Psi_L \equiv P_L\Psi, \qquad \Psi_R \equiv P_R\Psi. \tag{7}$$

1. Write the bilinears $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^\mu\Psi$ and $\bar{\Psi}\gamma^\mu\gamma^5\Psi$, then the Dirac Lagrangian, in terms of Ψ_L and Ψ_R. Show (again) that, if $m = 0$, the left and right spinors are decoupled.
2. Write the action of the transformations $U(1)_V$ et $U(1)_A$ on the L and R spinors. Show that we can interpret these transformations as two independent $U(1)_L$ et $U(1)_R$, each acting on Ψ_L et Ψ_R respectively.
3. Using the $L - R$ decomposition of the Dirac Lagrangian, show (again) that for $m = 0$ there is a full $U(1)_L \times U(1)_R$.

2 Spinor algebra

The Dirac matrices are defined in terms of the basic property :

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4 \tag{8}$$

where $g_{\mu\nu}$ is the Minkowski metric $diag(1, -1-1-1)$ and $\mathbf{1}_4$ is the identity matrix.

A basis for positive and negative frequency solutions of the Dirac equation is given by :

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi^s \\ \sqrt{p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu}\xi^s \\ -\sqrt{p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad s = 1, 2 \tag{9}$$

where ξ^s are the two-component spinors

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the expressions (9)

$$\sigma^\mu = (\mathbf{1}_2, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^i)$$

where σ^i are the Pauli matrices.

2.1 Traces of γ matrices

1. Without using the explicit representation of the γ-matrices, but only equation (8), show that

$$\begin{aligned} &\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad , \\ &\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \quad . \end{aligned}$$

   where the trace is over the spinor indices.
2. Deduce expressions for $\mathrm{Tr}\,(\not{a}\,\not{b})$ and $\mathrm{Tr}\,(\not{a}\,\not{b}\,\not{c}\,\not{d})$, where a_μ, b_μ etc. are 4-vectors.
3. Show that $\mathrm{Tr}\,(\gamma^\mu) = 0$

2.2 Spin sums

In what follows, α, β... are spinor indices, and run from 1 to 4, s, s'... run over the polarisation (1,2).

1. Show that

$$\bar{u}^s(p)u^{s'}(p) = 2m\delta^{ss'}, \quad \bar{v}^s(p)v^{s'}(p) = -2m\delta^{ss'}, \quad \bar{v}^s(p)u^{s'}(p) = \bar{u}^s(p)v^{s'}(p) = 0$$

2. Show that

$$\bar{u}^s(p)\gamma^\mu u^{s'}(p) = 2\delta^{ss'}p^\mu$$

3. Show that

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

Notice that in excerices 1 and 2 spinor indices are contracted, while in excercise 3 they are not.