M2 NPAC 2023/2024

QFT **problem set n°2**

QED

1 Dirac propagator

We define Dirac's propagator as :

$$S^F_{\alpha\beta}(x,y) = \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle \tag{1}$$

where the time-ordered product of two fermionic operators $A(x)$ and $B(y)$ is defined as follows :

$$TA(x)B(y) = \begin{cases} A(x)B(y) & x^0 > y^0 \\ -B(y)A(x) & x^0 < y^0 \end{cases}$$

Recall the plane wave expansions for free Dirac fields :

$$\psi_\alpha(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left[b^s_{\vec{p}} u^s_\alpha(p) e^{-i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})} + c^{s\dagger}_{\vec{p}} v^s_\alpha(p) e^{i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right] \tag{2}$$

where $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$, s runs over two polarizations, and $u(p)$ and $v(p)$ are positive and negative frequency spinors satisfying

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

The anti-commutation relations are :

$$\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}), \quad \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}) \tag{3}$$

and all other anti-commutators vanishing.

1. Show that the (unordered) fermion two-point functions can be written as

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} + m\delta_{\alpha\beta}\right) D(x-y),\ \langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle = \left(i\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu} - m\delta_{\alpha\beta}\right) D(y-x)$$

where

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} e^{-i(\omega_{\vec{p}}(x^0-y^0) - \vec{p}\cdot(\vec{x}-\vec{y}))}$$

2. Using Cauchy's theorem, rewrite $D(x-y)$ as a four-dimensional integral over momentum-space,

$$D(x-y) = \int_{C_+} \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{i}{p^2 - m^2} e^{-ip_\mu(x-y)^\mu} \tag{4}$$

where $p^\mu = (\omega, \vec{p})$ (with ω independent of $\vec{p}$), $p^2 \equiv p^\mu p_\mu$, and the integration over frequency is performed around an appropriate contour in the complex plane encircling the positive frequency pole on the real axis, $\omega_{\vec{p}} = +\sqrt{|\vec{p}|^2 + m^2}$.
What is the function one obtains by integrating instead around the contour C_- encircling the negative frequency pole, $\omega_{\vec{p}} = -\sqrt{|\vec{p}|^2 + m^2}$?

3. Use the results of the previous two points to show that the fermion time-ordered two-point correlator $S^F_{\alpha\beta}(x,y)$, defined in equation (1), is given by the expression

$$S^F_{\alpha\beta}(x,y) = i\int_{C_F} \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{\not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}}{p^2 - m^2} e^{-ip_\mu(x-y)^\mu}$$

the integral over frequency running along Feynman's contour C_F.

2 QED Cross Sections

The goal of this exercise is to calculate the unpolarized differential cross section for two simple QED processes, at tree level, in the center of mass frame. The result will be expressed as a function of the center of mass energy E_{CM} and the scattering angle θ (i.e. the angle between the outgoing particles and the incoming direction. The latter may be taken to be the z direction).

Recall that, for $2 \to 2$ scattering, the differential cross section in the center of mass is related to the amplitude by (cfr. TD3)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{A}|^2 \tag{5}$$

where $\mathcal{A}$ is the scattering amplitude and $\vec{p}_{i,f}$ are the initial and final momenta of one of the particles.

Recall that :

$$\begin{aligned}
&\mathrm{Tr}\,(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu} \quad , \\
&\mathrm{Tr}\,(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\,(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \quad .
\end{aligned}$$

where the trace is over the spinor indices.

Recall also the spin sum rules :

$$\bar{u}^s(p) u^{s'}(p) = 2m\delta^{ss'}, \quad \bar{v}^s(p) v^{s'}(p) = -2m\delta^{ss'}, \quad \bar{v}^s(p) u^{s'}(p) = \bar{u}^s(p) v^{s'}(p) = 0$$

$$\sum_{s=1}^{2} u^s_\alpha(p) \bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p) \bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

The Feynman rules for QED S-matrix elements can be found at the end of this sheet.

2.1 e^+e^- (Bhabha) Scattering

Consider the process

$$e^+ e^- \longrightarrow e^+ e^-$$

We want to compute the *unpolarized* cross section (i.e. averaged over initial spins and summed over final spins).

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Compute the square of the amplitude using the spin sum rules, and the corresponding differential cross section using equation (5). Show that, in the high-energy limit $E_{cm} \gg m_e$, one finds :

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left[u^2 \left(\frac{1}{s} + \frac{1}{t} \right)^2 + \left(\frac{t}{s} \right)^2 + \left(\frac{s}{t} \right)^2 \right]. \tag{6}$$

   where s, t, u are the Mandelstam variables *(Notice that, if we ignore the electron mass, then $s + t + u = 0$).*
4. Rewrite equation (6) in terms of $\cos\theta$ and the center-of-mass energy.

2.2 Pair annihilation into photons

Consider the process

$$e^+e^- \longrightarrow \gamma\,\gamma$$

in the center-of-mass frame. We want to compute the *unpolarized* cross section.

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Prove the *photon polarisation sum* rules :

$$\sum_{i=1}^{2}(\epsilon_\mu^i)^*\epsilon_\nu^i = -g_{\mu\nu} + \frac{1}{2E^2}(p_\mu\bar{p}_\nu + \bar{p}_\mu p_\nu) \tag{7}$$

   where $p_\mu = (E,\vec{p})$, $\bar{p}_\mu = (E,-\vec{p})$, ϵ^i with $i = 1,2$ are two transverse polarizations (i.e. orthogonal to both p_μ and $\bar{p}_\mu$ (Notice that if p_μ is a null vector, so is $\bar{p}_\mu$. For example, $p_\mu = (E,0,0,E)$, $\bar{p}_\mu = (E,0,0,-E)$, $\epsilon_\mu^1 = (0,1,0,0)$, $\epsilon_\mu^2 = (0,0,1,0)$.)
4. Show that the amplitude vanishes whenever the polarisation is along p_μ or $\bar{p}_\mu$. Deduce that we can substitute

$$\sum_{i=1}^{2}(\epsilon_\mu^i)^*\epsilon_\nu^i \to -g_{\mu\nu}$$

   when squaring the amplitude.
5. Compute the square of the amplitude using the result above *(make sure you first add the contribution from the two diagrams, then square)* and show that the corresponding differential cross section is :

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\frac{E}{p}\left[\frac{E^2+m^2+p^2\cos^2\theta}{m^2+p^2\sin^2\theta} - \frac{2m^4}{(m^2+p^2\sin^2\theta)^2}\right] \tag{8}$$

6. Show that, in the high-energy limit $E \gg m$, and for finite θ (i.e. for $\theta \gtrsim m/p$) equation (8) becomes

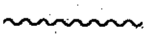
$$\frac{d\sigma}{d\cos\theta} \simeq \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)^2 \tag{9}$$

From Schwartz

13.1 QED Feynman rules

The Feynman rules for QED can be read directly from the Lagrangian just as in scalar QED. The only subtlety is possible extra minus signs coming from anticommuting spinors within the time ordering. First, we write down the Feynman rules, then derive the supplementary minus sign rules.

A photon propagator is represented with a squiggly line:



$$= \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{13.10}$$

Unless we are explicitly checking gauge invariance, we will usually work in Feynman gauge, $\xi = 1$, where the propagator is

$$= \frac{-ig_{\mu\nu}}{p^2 + i\varepsilon} \quad \text{(Feynman gauge)}. \tag{13.11}$$

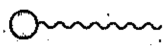
A spinor propagator is a solid line with an arrow:

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}. \tag{13.12}$$

The arrow points to the right for particles and to the left for antiparticles. For internal lines, the arrow points with momentum flow.

External photon lines get polarization vectors:

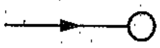
$$= \epsilon_\mu(p) \quad \text{(incoming)}, \tag{13.13}$$



$$= \epsilon^\star_\mu(p) \quad \text{(outgoing)}. \tag{13.14}$$

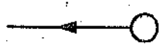
Here the blob means the rest of the diagram.

External fermion lines get spinors, with u spinors for electrons and v spinors for positrons.



$$= u^s(p), \tag{13.15}$$

$$= \bar{u}^s(p), \tag{13.16}$$



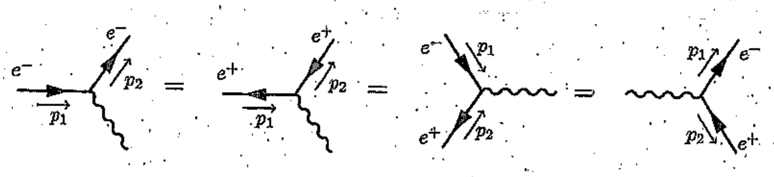
$$= \bar{v}^s(p), \tag{13.17}$$

$$= v^s(p). \tag{13.18}$$

External spinors are on-shell (they are forced to be on-shell by LSZ). So, for external spinors we can use the equations of motion:

$$(\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m) = 0, \tag{13.19}$$

$$(\not{p} + m)v^s(p) = \bar{v}^s(p)(\not{p} + m) = 0, \tag{13.20}$$



$$= -ie\gamma^\mu.$$

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