M2 NPAC 2023/2024

QFT **Problem set n°3**

1 Goldstone Modes and Residual Unbroken Symmetry

Consider a system of three complex scalars, ϕ_1, ϕ_2, ϕ_3 with Lagrangian :

$$\mathcal{L} = (\partial_\mu \phi_1^\dagger)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^\dagger)(\partial^\mu \phi_2) + (\partial_\mu \phi_3^\dagger)(\partial^\mu \phi_3) - V(\phi_1, \phi_2, \phi_3) \tag{1}$$

and potential :

$$V(\phi_1, \phi_2, \phi_3) = m^2 \left(|\phi_1|^2 + |\phi_2|^2\right) + \frac{\lambda}{4}\left(|\phi_3|^2 - \mu^2\right)^2 + v\left(\phi_1^* \phi_2 \phi_3^2 + \phi_1 \phi_2^* \phi_3^{*2}\right) \tag{2}$$

où

$$m^2 > 0,\ \mu^2 > 0,\ \lambda > 0,\ v \in \mathbb{R}$$

1. Show that the Lagrangian is invariant under the $U(1)_1 \times U(1)_2$ transformations :

$$\begin{array}{ll} U(1)_1 : & \phi_1 \to e^{i\theta_1}\phi_1,\ \phi_2 \to \phi_2,\ \phi_3 \to e^{i\theta_1/2}\phi_3 \\ U(1)_2 : & \phi_1 \to \phi_1,\ \phi_2 \to e^{i\theta_2}\phi_2,\ \phi_3 \to e^{-i\theta_2/2}\phi_3 \end{array}$$

   What is the symmetry of $\mathcal{L}$ if $v = 0$?
2. Write the Noether currents associated to $U(1)_1$ and $U(1)_2$.
3. Find the classical vacuum solutions
4. For each type of classical vacuum ?
   i. Identify the unbroken symmetry
   ii. Count and idenfity the Goldstone Bosons
   iii. Expand the Lagrangian to quadratic order in fluctuations around the classical vacuum, and obtain the particle spectrum (masses and charges).

2 Superconductivity as a Higgs Phenomenon

Consider scalar QED with a symmetry-breaking scalar potential :

$$\mathcal{L} = (D_\mu \phi^\dagger)(D^\mu \phi) + \frac{\mu^2}{2}\phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

where

$$\begin{array}{rcl} D_\mu \phi & \equiv & (\partial_\mu - ieA_\mu)\phi \\ F^{\mu\nu} & \equiv & \partial^\mu A^\nu - \partial^\nu A^\mu \end{array}$$

Considérer le cas statique où $\partial^0 \phi = \partial^0 \vec{A} = 0$ et $A_0 = 0$:

1. Show that the field equation for $\vec{A}$ has the form :

$$\vec{\nabla} \times \vec{B} = \vec{J} \quad \text{avec} \quad \vec{J} = ie[\phi^\dagger(\nabla - ie\vec{A})\phi - (\nabla + ie\vec{A})\phi^\dagger \phi]$$

2. Show that, in the presence of symmetry breaking, and in the classical approximation where $\phi = v = (\mu^2/\lambda)^{1/2}$, the current$\vec{J}$ takes the form :

$$\vec{J} = e^2 v^2 \vec{A} \tag{4}$$

   and therefore :

$$\nabla^2 \vec{B} = e^2 v^2 \vec{B} \tag{5}$$

   Equations (4) and (5) are called équation de *London Equation* and *Meissner effect equation*, respectively.

3. Show that, with broken symmetry, the resistivity ρ (defined by the relation $\vec{E} = \rho\vec{J}$) vanishes, leading to superconductivity.

3 Non-Abelian Higgs Mechanism

Consider a theory with an $SU(2)$ gauge symmetry, with gauge coupling g, spontaneously broken by a scalar multiplet in two different cases :

— a doublet of complex scalars

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

in the fundamental representation of $SU(2)$, with v.e.v :

$$\phi^\dagger \phi = v^2$$

— a triplet of real scalars

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

in the adjoint representation of $SU(2)$ (a.k.a. a 3-dimensional vector of $SO(3)$) with v.e.v. :

$$^t\Phi\,\Phi = v^2$$

We can suppose that the non-zero vev is obtained by minimizing a scalar potential of the form :

$$V = \frac{\lambda}{2}\left(X^2 - v^2\right) \qquad X \equiv \phi^\dagger\phi \; or \; {}^t\Phi\,\Phi$$

In each of the two cases above :

1. Determine the unbroken symmetry group ;
2. Define the the covariant derivative of the scalar field, and write explicitly its matrix form ;
3. Write the scalar field kinetic term and, by setting the scalar field to its vacuum value, determine the masses of the gauge fields.

Recall that the (hermitian) generators of $SU(2)$ are given by :

— Fundamental representation :

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

— Adjoint (3-dimensional vector) representation :

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$