2. MANY-BODY QUANTUM MECHANICS

One-body quantum systems

• Basic postulates (spin $\frac{1}{2}$ - isospin $\frac{1}{2}$ systems)

1) Allowed physical systems are represented as vector of a Hilbert space $\mathcal{H}_1$, i.e. $|\psi\rangle \in \mathcal{H}_1$

$\mathcal{H}_1$ can be decomposed as

$$\mathcal{H}_1 \equiv \mathcal{H}_{1,\text{space}} \otimes \mathcal{H}_{1,\text{spin}} \otimes \mathcal{H}_{1,\text{isospin}}$$

2) An observable A is obtained by the action of a self-adjoint operator $\hat{A}$ on $\mathcal{H}_1$

↓

The operators $\hat{\vec{r}}, \hat{\vec{p}}, \hat{\vec{s}}, \hat{\vec{t}}$ form an irreducible set

⇒ any operator $\hat{A} = \hat{A}(\hat{\vec{r}}, \hat{\vec{p}}, \hat{\vec{s}}, \hat{\vec{t}})$

↓

$\hat{\vec{r}}$ and $\hat{\vec{p}}$ obtained from classical counterpart

$$\hat{\vec{r}} \equiv \vec{r}\,\times \qquad (\to \text{act on } \mathcal{H}_{1,\text{space}})$$

$$\hat{\vec{p}} \equiv -i\hbar \vec{\nabla}$$

$\hat{\vec{s}}$ and $\hat{\vec{t}}$ purely quantum → expressed via Pauli matrices

$$\hat{\vec{s}} \equiv \frac{\hbar}{2} \vec{\sigma}_{\text{spin}} \qquad (\to \text{act on } \mathcal{H}_{1,\text{spin}})$$

$$\hat{\vec{t}} \equiv \frac{\hbar}{2} \vec{\sigma}_{\text{isospin}} \qquad (\to \text{act on } \mathcal{H}_{1,\text{isospin}})$$

↓
commutation relations

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

$$[\hat{s}_i, \hat{s}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{s}_k \qquad \text{where } i = x, y, z$$

$$[\hat{t}_i, \hat{t}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{t}_k$$

all other commutators = 0

↓
$\{\hat{\vec{r}}, \hat{s}_z, \hat{t}_z\}$ form a set of commutating operators

⇒ can be simultaneously diagonalised by the eigenbasis

$$|\vec{r}\sigma\tau\rangle = |\vec{r}\rangle \otimes |\sigma\rangle \otimes |\tau\rangle \quad \text{such as}$$

$$\begin{cases} \hat{r}_i |\vec{r}\sigma\tau\rangle = r_i |\vec{r}\sigma\tau\rangle \\ \hat{s}_z |\vec{r}\sigma\tau\rangle = \sigma |\vec{r}\sigma\tau\rangle \\ \hat{t}_z |\vec{r}\sigma\tau\rangle = \tau |\vec{r}\sigma\tau\rangle \end{cases}$$

3) The dynamical evolution of the system is governed by the time-dependent Schrödinger eq.

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{h} |\psi\rangle$$

where $\hat{h}$ is the Hamiltonian acting on $\mathcal{H}_1$

$$\hat{h} \equiv \hat{t}_{kin} + \hat{v} = \frac{\hat{\vec{p}}^2}{2m} + \hat{v}(\hat{\vec{r}}, \hat{\vec{p}}, \hat{\vec{s}}, \hat{\vec{t}})$$

4) Probability interpretation

$$|\langle \vec{r}\sigma\tau|\Psi\rangle|^2 = |\Psi(\vec{r}\sigma\tau)|^2$$

probability of finding a particle at $\vec{r}$ with σ and τ

5) Reduction of the wave packet

Given $\hat{A}|\phi_i\rangle = A_i|\phi_i\rangle$

and $|\Psi\rangle = \sum_i c_i |\phi_i\rangle$

measurement of the value A_K with probability $|c_K|^2$

$\Rightarrow$ state collapses to $|\Psi\rangle = |\phi_K\rangle$

Many-body quantum systems

- Extension of basic postulates

Basic postulates can be extended to systems with N particles

1) The space of allowed states $\mathcal{H}_N$ is given by tensor product of the one-particle Hilbert spaces associated with each of the N particles.

$$\mathcal{H}_N(1,2,\dots N) \equiv \mathcal{H}_1(1)\otimes\mathcal{H}_1(2)\otimes\dots\otimes\mathcal{H}_1(N)$$

2) The irreducible set of operators is the collection of the irreducible sets of each of the N particles

$$\hat{\vec{r}}_1, \hat{\vec{p}}_1, \hat{\vec{s}}_1, \hat{\vec{t}}_1, \dots \hat{\vec{r}}_N, \hat{\vec{p}}_N, \hat{\vec{s}}_N, \hat{\vec{t}}_N$$

drop the hat from now on

$\rightarrow$ operators associated to different particles commute

3) The dynamical Schrödinger eq. reads as

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle$$

where H denotes the N-body Hamiltonian

$$H = T + V = \underbrace{\sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m}}_{\text{kinetic energy operator}} + \underbrace{V(\vec{r}_1, \vec{p}_1, \vec{s}_1, \vec{t}_1, \ldots \vec{r}_N, \vec{p}_N, \vec{s}_N, \vec{t}_N)}_{\text{potential-energy operator}}$$

4) Probability interpretation

$$|\langle 1: \vec{r}_1 \sigma_1 \tau_1; \ldots ; N: \vec{r}_N \sigma_N \tau_N | \Psi \rangle|^2 = |\Psi(\vec{r}_1 \sigma_1 \tau_1, \ldots \vec{r}_N \sigma_N \tau_N)|^2$$

corresponds to the probability to find particle $i \in [1, N]$ at position $\vec{r}_i$, with spin projection σ_i and isospin projection τ_i

5) Reduction of the wave packet

→ Naturally extends from the one-body case

- Bases of $\mathcal{H}_N$

Given a basis $B_1 = \{|\alpha_1\rangle\}$ of the one-body Hilbert space $\mathcal{H}_1(1)$ the direct-product basis of $\mathcal{H}_N$ reads as

$$B_N \equiv \{ |1:\alpha_1; \ldots ; N:\alpha_N\rangle \equiv |1:\alpha_1\rangle \otimes |2:\alpha_2\rangle \otimes \ldots \otimes |N:\alpha_N\rangle \}$$

In general, one can expand any state $|\Psi\rangle \in \mathcal{H}_N$ as a linear combination of direct-product basis states

$$|\Psi\rangle = \sum_{\alpha_1 \dots \alpha_N} C_{\alpha_1 \dots \alpha_N} |1:\alpha_1\rangle \otimes \dots \otimes |N:\alpha_N\rangle$$

The particular case where only one term appears in the sum, i.e.

$$|\Psi\rangle = |1:\alpha_1\rangle \otimes \dots \otimes |N:\alpha_N\rangle$$

characterises a direct-product state (or simply product state).

In general, however, physical states can not be simply written as product states (i.e. a collection of independent particle) but are entangled.

- Operators

In many-body systems, there exist operators that act on more than one particle at a time.

A k-body operator (with $k \leq N$) can be decomposed as a sum of operators each acting non-trivially on k-body subspaces of $\mathcal{H}_N$

$$\mathcal{H}_k(i_1, \dots i_k) = \mathcal{H}_1(i_1) \otimes \mathcal{H}_1(i_2) \otimes \dots \otimes \mathcal{H}_1(i_k)$$

with $(i_1, \dots i_k) \in [1, N]$.

Notice that, in principle, any operator defined on $\mathcal{H}_N$ act on all N particles. However a, e.g., one-body operator has the form

$$\underbrace{O_1(1)}_{\text{acts "non-trivially" on one particle}} \otimes \underbrace{\mathbb{1}_1(2) \otimes \mathbb{1}_1(3) \otimes \dots \otimes \mathbb{1}_1(N)}_{\text{acts "trivially" on N-1 particles}}$$

One-body operator

A one-body operator F is defined via its action on a basis of $\mathcal{H}_N$ as

$$F : \mathcal{H}_N \to \mathcal{H}_N$$

$$|1{:}\alpha; 2{:}\beta; \dots\rangle \longrightarrow F|1{:}\alpha; 2{:}\beta; \dots\rangle$$

such that

$$F = \sum_{i=1}^{N} f(i)$$

with

$$f(i) : \mathcal{H}_1(i) \to \mathcal{H}_1(i)$$

$$|i{:}\alpha\rangle \to f(i)\,|i{:}\alpha\rangle$$

k-body operator

A k-body operator K is defined via its action on a basis of $\mathcal{H}_N$ as

$$K : \mathcal{H}_N \to \mathcal{H}_N$$

$$|1{:}\alpha; 2{:}\beta; \dots\rangle \longrightarrow K\,|1{:}\alpha; 2{:}\beta; \dots\rangle$$

such that

$$K = \frac{1}{k!} \sum_{i\neq j\neq \dots \neq l=1}^{N} k(i,j,\dots,l)$$

with $k(i,j,\dots l)$ acting non-trivially on $\mathcal{H}_k(i,j,\dots l)$

$$k(i,j,\dots l) : \mathcal{H}_k(i,j,\dots l) \longrightarrow \mathcal{H}_k(i,j,\dots l)$$

$$|i{:}\alpha; j{:}\beta; \dots l{:}\delta\rangle \longrightarrow k(i,j,\dots l)\,|i{:}\alpha; j{:}\beta; \dots l{:}\delta\rangle$$

- Operator representation

In practice, a k-body operator is defined in a chosen representation, i.e. through the set of its matrix elements in a given basis of $\mathcal{H}_K$.
Typically one starts with the eigenbasis of the irreducible operators

$$B_1(1) = \left\{ |1:\vec{r}\sigma\tau\rangle \right\} \Rightarrow \boxed{\langle 1:\vec{r}\sigma\tau|\, f(1)\, |1:\vec{r}'\sigma'\tau'\rangle \atop \underline{\text{operator matrix elements}}}$$

One might be interested in representing the operator in some other basis (e.g. harmonic osillator eigenfunctions for nuclear structure calculations)

$$B_1'(1) = \left\{ |1:\alpha\rangle \right\} \Rightarrow \langle 1:\alpha|\, f(1)\, |1:\beta\rangle$$

To do so, insert twice completeness relation on $\mathcal{H}_1(1)$

$$\mathbb{1}_1 = \int d\vec{r} \sum_{\sigma\tau} |1:\vec{r}\sigma\tau\rangle\langle 1:\vec{r}\sigma\tau|$$

and get

$$\langle 1:\alpha|\, f(1)\, |1:\beta\rangle = \int d\vec{r}_1\, d\vec{r}_1' \sum_{\substack{\sigma\tau \\ \sigma'\tau'}} \varphi_\alpha^*(\vec{r}_1\sigma\tau)\, f(\vec{r}_1\sigma\tau, \vec{r}_1'\sigma'\tau')\, \varphi_\beta(\vec{r}_1'\sigma'\tau')$$

where
$$\begin{cases} \varphi_\alpha(\vec{r}\sigma\tau) \equiv \langle 1:\vec{r}\sigma\tau | 1:\beta\rangle \\ f(\vec{r}\sigma\tau, \vec{r}'\sigma'\tau') \equiv \langle 1:\vec{r}\sigma\tau|\, f(1)\, |1:\vec{r}'\sigma'\tau'\rangle \end{cases}$$

Often an operator acts non-trivially only on $\mathcal{H}_1$, space, i.e. it is spin- and isospin-independent.

$$f(1) = f_{\text{space}}(1) \otimes \mathbb{1}_{\text{spin}}(1) \otimes \mathbb{1}_{\text{isospin}}(1)$$

Then $\langle 1:\vec{r}_1\sigma\tau|\, f(1)\, |1:\vec{r}\,'\sigma'\tau'\rangle = \underbrace{\langle 1:\vec{r}_1|\, f_{space}(1)\, |1:\vec{r}_1\,'\rangle}_{= f(\vec{r}_1, \vec{r}_1\,')} \delta_{\sigma\sigma'}\delta_{\tau\tau'}$

Matrix elements in basis $\mathcal{B}_1^1(1) = \{|1:\alpha\rangle\}$ also simplify

$$\langle 1:\alpha|\, f(1)\, |1:\beta\rangle = \int d\vec{r}_1\, d\vec{r}_1\,' \sum_{\sigma\tau} \varphi_\alpha^*(\vec{r}_1\sigma\tau)\, f(\vec{r}_1, \vec{r}_1\,')\, \varphi_\beta(\vec{r}_1\sigma\tau)$$

The same steps can be easily generalised to the case of a k-body operator.

- Identical particles

= carrying the same intrinsic quantum numbers (mass, charge, spin, isospin)

In the case of a system of N identical particles, not all states of $\mathcal{H}_N$ are physically allowed.

$\Rightarrow$ Two additional postulates are then needed. These two postulates apply differently to bosons and fermions.

Some notions must be introduced before stating the additional postulates.

Symmetric group

Formed by the permutation of N elements, with the following properties

i) The group contains $N!$ elements p called permutations of $(1, 2, \dots N)$. The set of all permutations is $\mathcal{P}$

ii) Any permutation $p \in \mathcal{P}$ can be decomposed as a product of elementary transpositions t_{ij} defined via

$$t_{ij}(i) = j \qquad t_{ij}(j) = i \qquad t_{ij}(k) = k \quad \text{if } k \neq i, j$$

E.g. $p: (1,2,3) \to (3,1,2) \quad \Rightarrow \quad p = t_{23} t_{13}$

iii) One associates a number $\pi_p = \pm 1$ to each $p \in \mathcal{P}$, called signature, such that

$$\pi_{pp'} = \pi_p \pi_{p'}$$

$$\pi_{\text{identity}} = +1$$

$$\pi_{t_{ij}} = -1$$

Given the set of permutations $\mathcal{P}$, $N!$ operators P_p are defined on $\mathcal{H}_N$ through the action on the basis states

$$P_p |1:\alpha_1 \dots N:\alpha_N\rangle \equiv |1:\alpha_{p(1)} \dots N:\alpha_{p(N)}\rangle$$

$$\equiv |p^{-1}(1):\alpha_1 \dots p^{-1}(N):\alpha_N\rangle$$

E.g., transpositions $P_{ij} \equiv P_{t_{ij}}$

$$P_{ij} |1:\alpha_1 \dots i:\alpha_i \dots j:\alpha_j \dots\rangle = |1:\alpha_1 \dots i:\alpha_j \dots j:\alpha_i \dots\rangle$$

<u>Symmetrisation and antisymmetrisation operators</u>

Let us introduce the following operators acting on $\mathcal{H}_N$

$$S \equiv \frac{1}{N!} \sum_{p \in \mathcal{P}} P_p \qquad \text{symmetrisation}$$

$$A = \frac{1}{N!} \sum_{p \in \mathcal{P}} \pi_p P_p \qquad \text{antisymmetrisation}$$

with the properties

$$S^\dagger = S \qquad \mathcal{A}^\dagger = \mathcal{A}$$

$$P_p S = S \qquad P_p \mathcal{A} = \pi_p \mathcal{A}$$

$$S^2 = S \qquad \mathcal{A}^2 = \mathcal{A}$$

$$S\mathcal{A} = \mathcal{A}S = 0$$

Example: for $N=2$

$$\begin{cases} S = \frac{1}{2}\left(\mathbb{1}_2 + P_{12}\right) \\ \mathcal{A} = \frac{1}{2}\left(\mathbb{1}_2 - P_{12}\right) \end{cases}$$

Additional postulates

6) The N-body Hilbert space $\mathcal{H}_N$ must be reduced to the subspace of physically accessible states

$$\mathcal{H}_N^B \quad \text{for bosons}$$

$$\mathcal{H}_N^F \quad \text{for fermions}$$

i.e. states that are respectively symmetric and antisymmetric with respect to the exchange of <u>any two particles</u>

(Let us focus on fermions from now on)

A basis $\mathcal{B}_N^F$ of $\mathcal{H}_N^F$ is made of normalised and (fully) antisymmetric product states obtained from standard product states via

$$|\alpha_1 \ldots \alpha_N\rangle \equiv \sqrt{N!}\,\mathcal{A}\,|1:\alpha_1 \ldots N:\alpha_N\rangle$$

$$= \frac{1}{\sqrt{N!}} \sum_{p\in P} \pi_p \,|p(1):\alpha_1 \ldots p(N):\alpha_N\rangle$$

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} |1:\alpha_1\rangle & \ldots & |N:\alpha_1\rangle \\ \vdots & \ddots & \vdots \\ |1:\alpha_N\rangle & \ldots & |N:\alpha_N\rangle \end{vmatrix}$$

$$\equiv |\Phi\rangle$$

SLATER DETERMINANTS

Slater determinants have the property

$$\boxed{P_{ij}\,|\Phi\rangle = -\,|\Phi\rangle} \qquad \forall\, i \neq j$$

As a result of the antisymmetrisation, one can not say anymore that particle 1 is in state α_1 etc..

$\Rightarrow$ Intrinsic correlation between the N particles, fingerprint of Pauli principle

7) The space of physical operators must be also reduced

$\rightarrow$ Operators must be symmetric under the exchange of any pair of particles

$$\forall\,(i,j) \in (1,N) \qquad V(1,\ldots i,\ldots j,\ldots N) = V(1,\ldots j,\ldots i,\ldots N)$$

One nucleon states

- Eigenbasis of position

$$|\vec{r}\sigma\tau\rangle \equiv |\vec{r}\rangle \otimes |\sigma\rangle \otimes |\tau\rangle$$
$$\cap \qquad \cap \qquad \cap$$
$$\mathcal{H}_{1,\text{space}} \quad \mathcal{H}_{1,\text{spin}} \quad \mathcal{H}_{1,\text{isospin}}$$

- Eigenbasis of momentum $|\vec{p}\sigma\tau\rangle$

with $\langle \vec{r}\sigma\tau | \vec{p}\sigma'\tau'\rangle = \frac{1}{(2\pi)^{3/2}} e^{\frac{i}{\hbar}\vec{r}\vec{p}} \delta_{\sigma\sigma'}\delta_{\tau\tau'}$

- Eigenbasis of orbital momentum

$$\vec{\ell} = \vec{r} \times \vec{p} \qquad \Rightarrow \qquad [\ell_i, \ell_j] = i\hbar \sum_k \varepsilon_{ijk} \ell_k$$

One can show that $\vec{\ell}^2$ commutes with all components ℓ_i

$\Rightarrow$ Introduce eigenstates of $\vec{\ell}^2$ and e.g. ℓ_z

$$\begin{cases} \vec{\ell}^2 |\ell m_\ell\rangle = \hbar^2 \ell(\ell+1) |\ell m_\ell\rangle \\ \ell_z |\ell m_\ell\rangle = \hbar m_\ell |\ell m_\ell\rangle \end{cases} \quad \text{with} \quad \begin{cases} \ell \in \mathbb{N}, \; m_\ell \in \mathbb{Z} \\ |m_\ell| \leq \ell \end{cases}$$

In spherical coordinates $|\vec{r}\rangle = |r\theta\varphi\rangle$ $\vec{\ell}$ acts only on the angular part $|\theta\varphi\rangle$ and one can introduce spherical harmonics

$$Y_\ell^{m_\ell}(\theta,\varphi) \equiv \langle \theta\varphi | \ell m_\ell\rangle$$

(eigenfunctions of $\vec{\ell}^2$ in the angular subspace)

- Eigenbasis of total angular momentum

$$\vec{j} = \vec{\ell} + \vec{s}$$

Obeys usual angular momentum commutation relations
One can define common eigenbasis to $\vec{\ell}^2$ $\vec{s}^2$ $\vec{j}^2$ and j_z

$$|(\ell s) j m\rangle = |(\ell \tfrac{1}{2}) j m\rangle$$

usually omit subscript

$$\begin{cases} \vec{j}^2 |(\ell \tfrac{1}{2}) j m\rangle = \hbar^2 j(j+1) |(\ell 1/2) j m\rangle \\ j_z |(\ell 1/2) j m\rangle = \hbar m |(\ell 1/2) j m\rangle \end{cases} \quad \text{with} \quad \begin{cases} 2j \in \mathbb{N}^*, \ 2m \in \mathbb{Z} \\ |m| \leq j \end{cases}$$

These states can be defined from $|\ell m_\ell\rangle$ and $|\tfrac{1}{2} \sigma\rangle$ as

$$|(\ell 1/2) j m\rangle = \sum_{m_\ell \sigma} \langle \ell 1/2\, m_\ell \sigma | j m\rangle \left[|\ell m_\ell\rangle \otimes |1/2\, \sigma\rangle \right]$$

Clebsch-Gordan coefficient

Two-nucleon states

- Eigenbasis of position

Start with direct product basis of $\mathcal{H}_2$

$$|1: \vec{r}_1 \sigma_1 \tau_1 ; 2: \vec{r}_2 \sigma_2 \tau_2\rangle \equiv |1: \vec{r}_1 \sigma_1 \tau_1\rangle \otimes |2: \vec{r}_2 \sigma_2 \tau_2\rangle$$

and get antisymmetrised basis of $\mathcal{H}_2^{\mp}$

$$|\vec{r}_1 \sigma_1 \tau_1 ; \vec{r}_2 \sigma_2 \tau_2\rangle = \sqrt{2}\, A_{12} |1: \vec{r}_1 \sigma_1 \tau_1 ; 2: \vec{r}_2 \sigma_2 \tau_2\rangle$$

$$= \frac{1}{\sqrt{2}} \left[|1: \vec{r}_1 \sigma_1 \tau_1 ; 2: \vec{r}_2 \sigma_2 \tau_2\rangle - |1: \vec{r}_2 \sigma_2 \tau_2 ; 2: \vec{r}_1 \sigma_1 \tau_1\rangle \right.$$

These states are globally antisymmetric

under the exchange of all (i.e. spatial, spin, isospin) coordinates at once

- Centre-of-mass decoupling

Let us introduce c.o.m. and relative coordinates

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \qquad \vec{p} = \frac{\vec{p}_1 - \vec{p}_2}{2}$$

(Nuclear interaction only depends on relative motion of the two nucleons)

- Eigenbasis of orbital angular momentum

$$\vec{L}_{tot} = \vec{\ell}_1 + \vec{\ell}_2 = \underbrace{\vec{R} \times \vec{P}}_{\equiv \vec{L}_{c.o.m.}} + \underbrace{\vec{r} \times \vec{p}}_{\equiv \vec{L}}$$

One then introduces the eigenbasis $|L M_L\rangle$, with corresponding wave functions

$$Y_L^{M_L}(\theta, \varphi) \equiv \langle \theta \varphi | L M_L \rangle$$

Under the exchange of two nucleons

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \longrightarrow -\vec{r} \Rightarrow \begin{cases} \theta \to \pi - \theta \\ \varphi \to \varphi + \pi \end{cases}$$

One has

$$Y_L^{M_L}(\pi - \theta, \varphi + \pi) = (-1)^L \, Y_L^{M_L}(\theta, \varphi)$$

symmetric or antisymmetric depending on L

- Eigenbasis of spin

Define total (two-nucleon) spin operator $\vec{S} = \vec{s}_1 + \vec{s}_2$ and associated basis $|S M_S\rangle$. By expressing $|S M_S\rangle$ in terms of direct-product states

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\Big[|1{:}\uparrow\; 2{:}\downarrow\rangle - |1{:}\downarrow\; 2{:}\uparrow\rangle \Big]$$

$$|1\,1\rangle = |1{:}\uparrow\; 2{:}\uparrow\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\Big[|1{:}\uparrow\; 2{:}\downarrow\rangle + |1{:}\downarrow\; 2{:}\uparrow\rangle \Big]$$

$$|1\,-1\rangle = |1{:}\downarrow\; 2{:}\downarrow\rangle$$

one immediately sees that the spin singlet $|00\rangle$ is anti-symmetric under the exchange of particle 1 and 2, wile the spin triplet $|1\, M_S\rangle$ is symmetric. $\Rightarrow$ factor $(-1)^{S-1}$

The same property can be deduce by studying the action of the spin-exchange operator

$$P_\sigma \equiv \vec{S}^{\,2} - 1 \qquad (\rightarrow \text{exercise})$$

- Eigenstates of isospin

Exactly analogous to the spin with $\uparrow \longleftrightarrow n$ and $\downarrow \longleftrightarrow p$ (Notice that usually the opposite convention is adopted in particle physics)

$\Rightarrow$ Factor $(-1)^{T-1}$ under the exchange of the two nucleons

- Eigenbasis of total angular momentum

In the c.o.m. frame one can couple $\vec{L}$ and $\vec{S}$ to $\vec{J}$ giving rise to a basis of $\mathcal{H}_{2,\text{space}} \otimes \mathcal{H}_{2,\text{spin}}$

$$\vec{J} = \vec{L} + \vec{S} \Rightarrow |(LS)JM\rangle = \sum_{M_L M_S} (L S M_L M_S | JM) \left[|LM\rangle \otimes |SM_S\rangle \right]$$

$$|L-S| \leq J \leq L+S$$

$$\Downarrow$$

$$|J-S| \leq L \leq J+S$$

Usually we employ the spectroscopic notation ${}^{(2S+1)}[L]_J$, where $[L]$ denotes a letter corresponding to a value of L

L	$[L]$
0	S
1	P
2	D
3	F
4	G
⋮	⋮

"partial waves"

$$\begin{cases} J=0 \Rightarrow {}^1S_0,\ {}^3P_0 \\ J=1 \Rightarrow {}^1P_1,\ {}^3S_1,\ {}^3P_1,\ {}^3D_1 \\ \vdots \end{cases}$$

- Spin-isospin channels

One obtains a complete basis of $\mathcal{H}_2$ through the tensor product

$$|(LS)JM\rangle \otimes |T M_T\rangle$$

This must be antisymmetric under the exchange of the two nucleons, i.e., using the properties under exchange derived so far,

$$(-1)^L (-1)^{S-1} (-1)^{T-1} = -1 \Rightarrow (-1)^{L+S+T} = -1 \Rightarrow L+S+T \ \underline{\text{odd}}$$