3. NUCLEAR INTERACTIONS

Generalities

- Relevant degrees of freedom : structureless nucleons
- Non-relativistic framework (+ relativistic corrections)

  Nucleon dynamics in nuclei is to a large extent non-relativistic

$$\frac{|\vec{p}|}{m} \approx \frac{200 \text{ MeV}}{939 \text{ MeV}} \Rightarrow \left(\frac{v}{c}\right)^2 < 0.1$$

- Interaction is istanteneous, potential is time independent
- Strong + weak + EM accounted for
- Here we consider the ab initio Hamiltonian (as opposed to effective)

AB INITIO	EFFECTIVE
in vacuum	in medium
nucleus independent	nucleus dependent
links nuclei to QCD	focuses on phenomenological description
$H\lvert\Psi^A\rangle = E^A\lvert\Psi^A\rangle$	$H^A_{eff}\lvert\Psi^A_{eff}\rangle = E^A\lvert\Psi^A_{eff}\rangle$

- General form

$$H = \sum_{i=1}^{A}\frac{p_i^2}{2M} + \frac{1}{2}\sum_{i\neq j=1}^{A} V(i,j) + \frac{1}{3!}\sum_{i\neq j\neq k=1}^{A} W(i,j,k) + \dots$$

which is the form of these operators?

how many do we need to include?

- Over the years, three different approaches have been developed to model nuclear interactions

  I) Symmetry-driven

  II) Physics-driven

  III) Effective field theory

Symmetry-driven approach

- Most general form of two-nucleon interaction

$$V(1,2) = V(\vec{r}_1, \vec{p}_1, \vec{s}_1, \vec{t}_1, \vec{r}_2, \vec{p}_2, \vec{s}_2, \vec{t}_2)$$

idea: use known symmetries to constrain the operatorial form

- Nuclear interactions invariant under
  - exchange of two nucleons
  - translations in space
  - translations in time
  - rotations
  - Galilean boost
  - parity
  - time-reversal
  - isospin (approximate)

Potential can be written as a part that is invariant under isospin rotations

$$\tilde{V}(p,p) = \tilde{V}(p,n) = \tilde{V}(n,n)$$

$$V = \tilde{V} + V^{ISB}$$

+ small isospin-symmetry-breaking corrections

$$V^{ISB}(p,p) \neq V^{ISB}(p,n) \neq V^{ISB}(n,n)$$

The most general operator structure consistent with all symmetry constraints can be then derived.

Two examples

1) isospin operators

Potential must be a scalar with respect to isospin rotations

$\Rightarrow$ can depend only on scalar product

$$\Rightarrow V(\ldots, \vec{t}_1, \vec{t}_2) = F(\ldots, \vec{t}_1 \cdot \vec{t}_2)$$

$$= F_0(\ldots) + F_1(\ldots)\, \vec{t}_1 \cdot \vec{t}_2 + F_2(\ldots)\,(\vec{t}_1 \cdot \vec{t}_2)^2 + \ldots$$

One can prove that

$$(\vec{t}_1 \cdot \vec{t}_2)^2 \,|T M_T\rangle = (3 - 2\vec{t}_1 \cdot \vec{t}_2)\,|T M_T\rangle \qquad \to \text{exercise}$$

$$\Rightarrow F_0(\ldots) + F_1(\ldots)\, \vec{t}_1 \cdot \vec{t}_2$$

ii) linear dependence on spin operators

Since the potential is a scalar under spatial rotations, terms linear in $\vec{s}$ must appear under the form of a scalar product

$$V(\ldots \vec{s}_1, \vec{s}_2) = F(\ldots, \underbrace{\vec{a} \cdot \vec{s}_1 + \vec{b} \cdot \vec{s}_2}_{\text{invariance under permutations } 1 \leftrightarrow 2 \text{ implies } \vec{a} = \vec{b}})$$

$$\Rightarrow \vec{a} \cdot (\vec{s}_1 + \vec{s}_2) = \vec{a} \cdot \vec{S}$$

possibilities : $\underbrace{\vec{r}, \vec{p}}_{\text{wouldn't be invariant under parity}}, \vec{L}, \vec{S}$

$\vec{S}$ — would be trivial (a constant)

$\vec{L}$ — only non-trivial possibility

At the end, the most general operator structure reads

$$V(1,2) = V^{0} + V^{s}\,\vec{s}_1\cdot\vec{s}_2 + V^{t}\,\vec{t}_1\cdot\vec{t}_2 + V^{st}\,(\vec{s}_1\cdot\vec{s}_2)(\vec{t}_1\cdot\vec{t}_2)$$

with

$$V^{i} = \sum_{k=1}^{5} f_k^{i}\left(\vec{r}^{\,2}, \vec{p}^{\,2}, \vec{L}^{\,2}\right) O_k \qquad (i = 0, s, t, st)$$

and

$$O_k = \begin{cases} \mathbb{1} & \\ \vec{L}\cdot\vec{S} & \text{spin-orbit} \\ S_{12}^{r} \equiv 3(\vec{s}_1\cdot\hat{r})(\vec{s}_2\cdot\hat{r}) - \vec{s}_1\cdot\vec{s}_2 & \text{tensor} \\ S_{12}^{p} \equiv 3(\vec{s}_1\cdot\hat{p})(\vec{s}_2\cdot\hat{p}) - \vec{s}_1\cdot\vec{s}_2 & \text{tensor} \\ Q_{12} \equiv \frac{1}{2}\left[(\vec{s}_1\cdot\vec{L})(\vec{s}_2\cdot\vec{L}) + (\vec{s}_2\cdot\vec{L})(\vec{s}_1\cdot\vec{L})\right] & \text{quadratic spin-orbit} \end{cases}$$

where $\hat{x} \equiv \dfrac{\vec{x}}{|\vec{x}|}$.

- Tensors are defined with an additional $\vec{s}_1\cdot\vec{s}_2$ such that the average over the angles of $\vec{r}$ and $\vec{p}$ gives zero
- Q_{12} is called quadratic spin-orbit because it can be reexpressed as a function of $(\vec{L}\cdot\vec{S})^2$, $\vec{L}\cdot\vec{S}$ and $\vec{L}^{\,2}$

→ This approach was followed to develop the Argonne-V18 potential model (AV18)

→ Three-nucleon forces? One could follow the same strategy, but it gets very complicated

Physics-driven approach

First models date back to the 1930's

What was known at the time?

- Coulomb interaction between charged particles (infinite range)
- Nuclear interaction is short-ranged (~ 2 fm)

→ • All nuclei are compact
• Nuclear force negligible for atoms
• First scattering experiments

Yukawa 1935

↓

Idea: nuclear force mediated by massive spin-0 boson

→ "Mesotron"

(later → pion)

range ~ 2 fm ⇒ $m \sim 100$ MeV

→ $\sim$ Compton wavelenght of exchanged boson

Yukawa potential

$$V(r) \propto \frac{e^{-mr}}{r}$$

- One-pion exchange describes long-range attraction
- Generates tensor and $\vec{\tau}\cdot\vec{\tau}$ structures
- However, shorter-range part unaccounted for

→ Multi-pion exchange? Doesn't work.

→ 1960s: discovery of heavier mesons

↓

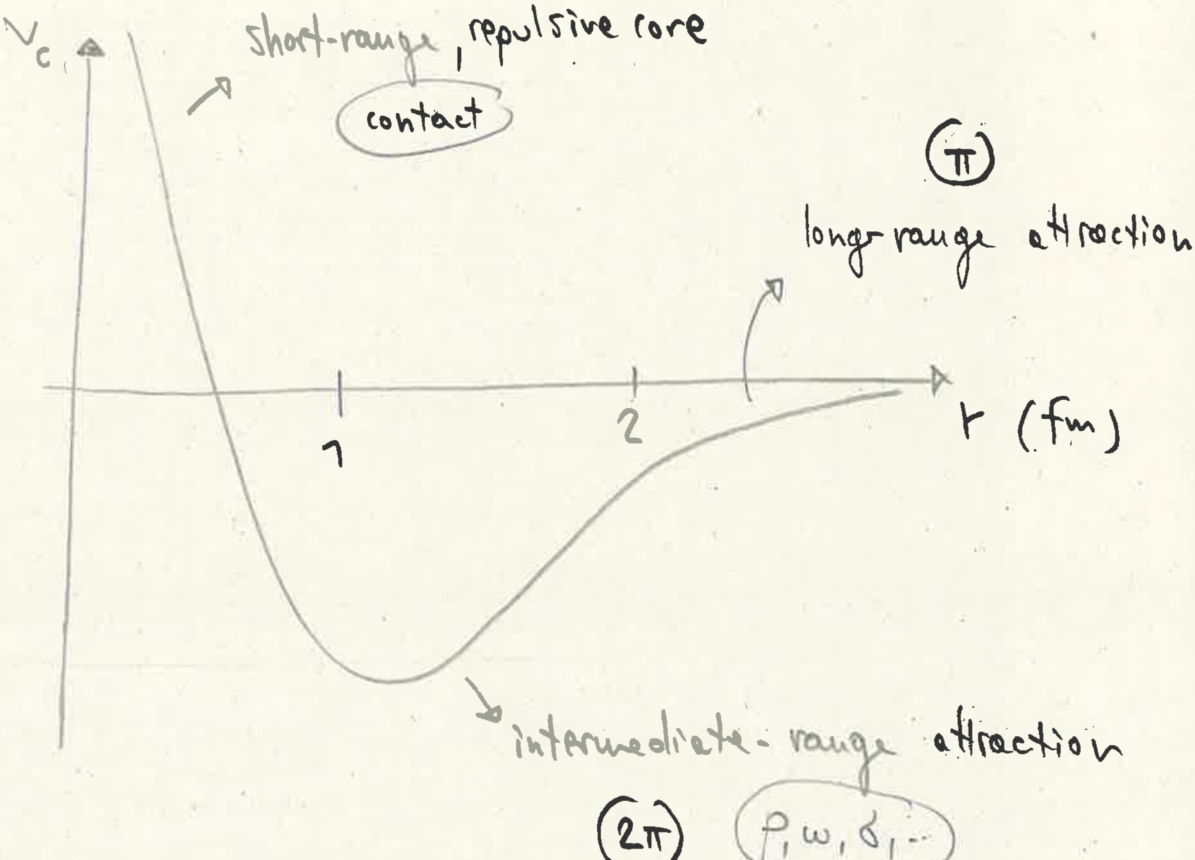
Can model ranges smaller than $\frac{1}{m_\pi}$

- ρ, ω, σ mesons generate different operator structures
- Very precise NN potentials constructed in this way

$\boxed{\text{One-boson exchange (OBE) potentials}}$

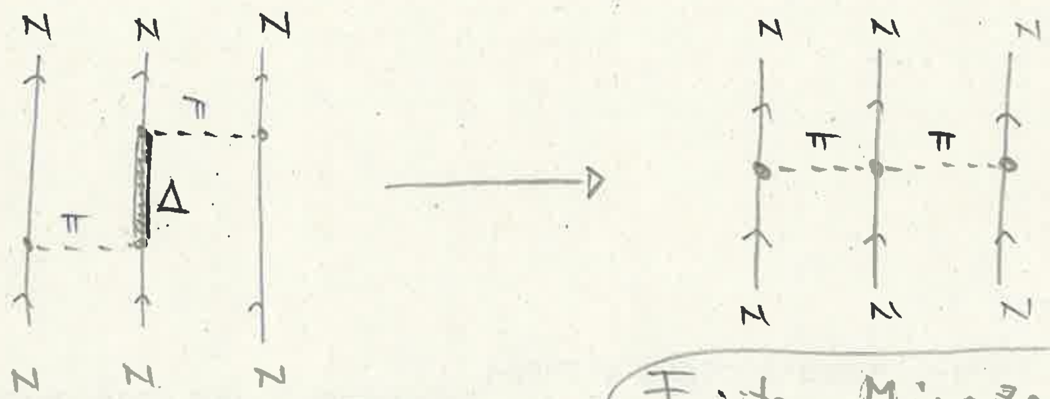
1980's $\rightarrow$ $\chi^2 \sim 2$

1990's $\rightarrow$ $\chi^2 \sim 1$



- Three-nucleon forces?

Certain processes e.g. involving nucleonic excitations can not be described in terms of NN interactions



Fujita - Miyazawa 1957

Effective-field theory approach (EFT)

- Symmetry-driven and physic-driven potentials very successful, but phenomenological models

  no systematic way to improve

  On the contrary, effective (field) theories provide a more systematic framework

- EFT introduced in nuclear physics in the 1990's (Weinberg, van Kolck, ...) → today: most used approach

- EFT principles

  1) Separation of scales
     → Use separation of energy scales to define d.o.f. and expansion parameter ε

     $$\varepsilon = \frac{Q}{M}$$

     Q: typical scales at play in the considered physical system

     M: high-energy scale

  2) Write all possible interaction terms (Lagrangian) between d.o.f. allowed by symmetries of underlying theory

  3) Order by size all these terms
     → Scaling ε^{ν} determine ν ⇒ "Power counting"

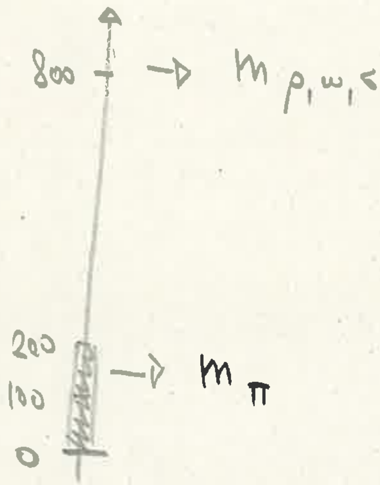
4) Truncate the expansion at a given order and compute observables

→ Unknown parameters (coupling constants) have to be fixed (underlying theory or experiment)

→ Order-by-order convergence has to be verified

- Application to nuclear forces

1) → Typical nucleon momenta $Q \sim 100 \div 200$ MeV



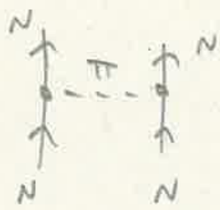
scale separation

$$\frac{Q}{M} \sim \frac{1}{3} \text{ or } \frac{1}{4}$$

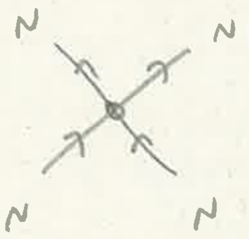
⇒ Degrees of freedom explicit in theory : N, π

2) → Interactions of two types

pion exchanges



contact interactions



3) → Most used ordering is Weinberg power counting

General equation for k-nucleon diagram

$$\nu = 2K - 4 + 2L + \sum_i \Delta_i$$

vertices

where

$$\begin{cases} L = \text{number of loops} \\ \Delta_i = d_i + \dfrac{n_i}{2} - 2 \end{cases}$$

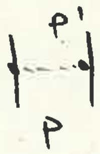
derivatives

nucleon fields

4) → At lowest order (= "leading order" (LO)) two terms

pure contact 

$$V^{(0)}_{ct}(p,p') = C_S + C_T\, \vec{S}_1 \cdot \vec{S}_2$$
– central

one-pion exchange 

$$V^{(0)}_{1\pi}(p,p') = -\frac{g_A^2}{4f_\pi^2}\, \vec{t}_1 \cdot \vec{t}_2\, \frac{\vec{S}_1\cdot\vec{q}\;\; \vec{S}_2\cdot\vec{q}}{q^2 + m_\pi^2}$$

tensor

↓

Higher orders and many-nucleon sectors (→ see slides)

Experimental constraints: NN scattering

Reaction types:

- np scattering is the easiest: neutron beams on hydrogen targets

- pp scattering : technically easy, but EM interaction needs to be substracted (beyond Coulomb, radiative corrections, ...)

- nn scattering : only indirect information (no n targets)
  - nd scattering (then substract np component)
  - two neutrons in final state, e.g. $n + d \to n + n + p$
  - comparison between different reactions

$$^3He + {}^3He \longrightarrow \alpha + p + p$$
$$^3He + {}^3H \longrightarrow \alpha + p + n$$
$$^3H + {}^3H \longrightarrow \alpha + n + n$$

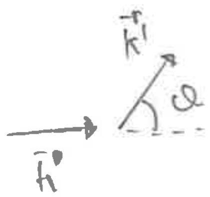
For energies beyond the pion production threshold ($\sim 2 m_\pi$ in lab frame) an inelastic component appears in the potential.

Cross section and partial-wave expansion

Relevant quantities

usually independent of φ

- differential cross section $\frac{d\sigma}{d\Omega}(E, \Theta)$
- total cross section $\sigma(E)$



$\to$ elastic scattering $\Rightarrow |\vec{k}| = |\vec{k}'|$ and $E = \frac{k^2}{2\mu}$ with $\mu = \frac{M}{2}$ reduced mass

Wave function at $r \to \infty$

$$\Psi_k(\vec{r}) \xrightarrow[r \to \infty]{} \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(k, \vartheta)\, \frac{e^{ikr}}{r}}_{\text{outgoing}}$$

$f(k,\theta)$ is the scattering amplitude $\Rightarrow$ $\boxed{\dfrac{d\sigma(k,\theta)}{d\Omega} = |f(k,\theta)|^2}$

By expanding $\psi_E(\vec{r})$ in partial waves one obtains

$$\psi_k(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, \frac{\underbrace{(-1)^{l+1} e^{-ikr}}_{\text{incoming}} + \underbrace{S_l(k)\, e^{ikr}}_{\text{outgoing}}}{2ikr}$$

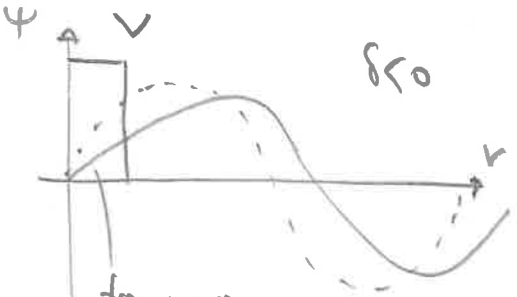
$S_l(k) = 1 + 2ik\, f_l(k)$ is the scattering matrix, usually parametrised as

$$S_l(k) = e^{2i\delta_l(k)} \qquad (\text{S-matrix is unitary})$$

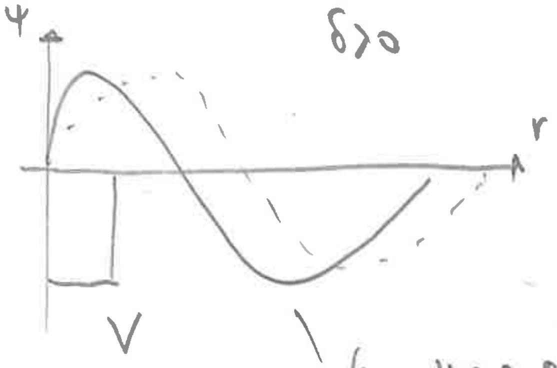
Then

$$\boxed{f_l(k) = \frac{S_l(k)-1}{2ik} = \frac{e^{i\delta_l(k)}}{k}\sin\delta_l(k) = \frac{1}{k\cot\delta_l(k) - ik}}$$

Outside the range of the interaction $(r\to\infty)$ the potential leads to a change in the phase of the outgoing wave



free wave gets pulled out by repulsive V



free wave gets pulled in by attractive V

$$\Rightarrow \sigma_{tot}(E) = \int d\Omega\, \frac{d\sigma(k,\theta)}{d\Omega} = \int d\Omega\, |f(k,\theta)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

Low-energy scattering

- Schwinger showed that $k^{2l+1} \cot \delta_l(k)$ has an analytic expansion

$$l=0 \rightarrow k \cot \delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_e k^2 - P^2 r_e^3 k^4 + \mathcal{O}(k^6)$$

$$l=1 \rightarrow k^3 \cot \delta_1(k) = -\frac{3}{a_p^3} + \mathcal{O}(k^2)$$

Effective range expansion

In the limit $k \rightarrow 0$ only $l=0$ scattering is present and

$$f_0(k) \rightarrow -a \Rightarrow \frac{d\sigma}{d\Omega} = a^2 \Rightarrow \sigma_{tot} = 4\pi a^2$$

in this limit a is the radius of an absorption disc equivalent to $\sigma_{tot}(0)$

→ effective range represents first correction ~ range of the interaction

- Scattering length → useful to expand $\cot \delta_0(k) \Rightarrow$ get $\delta_0(k) = -ka$

$a > 0$ bound state (see inset)
- a small → strong attraction
- a large → weak attraction

$a < 0$ no bound state
- a small → repulsion
- a large → weak repulsion / weak attraction

$$f_0(k) \xrightarrow[k \to 0]{} \frac{1}{-\frac{1}{a_0} - ik}$$

pole at $k = \frac{i}{a}$

$$\Rightarrow e^{ikr} \rightarrow e^{-\frac{r}{a}}$$

• Case of nucleon-nucleon scattering

→ At low energies, only $L=0$ ⇒ two possible channels:

spin singlet $^1S_0$ (s) and spin triplet $^3S_1$ (t)

$$a_{s\,(np)} \simeq -23.7 \text{ fm} \qquad r_{s\,(np)} \simeq 2.73 \text{ fm}$$

$$a_{t\,(np)} \simeq +5.4 \text{ fm} \qquad r_{t\,(np)} \simeq 1.75 \text{ fm}$$

$$a_{s\,(pp)} \simeq -17.1 \text{ fm} \qquad r_{s\,(pp)} \simeq 2.84 \text{ fm}$$

(after correcting for Coulomb)

$$a_{s\,(nn)} \simeq -16.6 \text{ fm} \qquad r_{s\,(nn)} \simeq 2.66 \text{ fm}$$

→ Charge symmetry realised to a very good extent

→ Charge independence more approximate

→ Bound state in $^3S_1$ channel (n-p) ⇒ deuteron

→ Scattering length large also in $^1S_0$ channel

⇒ almost-bound state (resonance) of two neutrons — di-neutron

(two protons excluded because of Coulomb)

Phase shifts and high-energy scattering

Phase shifts are used to parametrise scattering data based on a partial wave analysis

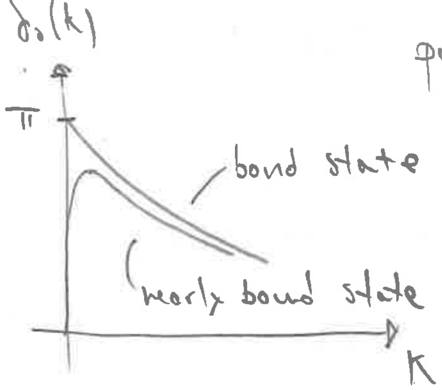
→ see e.g. Nijmegen partial wave analysis nn-online.org/NN

Discussion of $T=1$ and $T=0$ phase shifts (see figures)

- Presence of a bound state in $^3S_1$ channel (consistent with scattering length) and a nearly-bound state in $^1S_0$

Levinson's theorem $\delta_l(k=0) = n_l \pi$

number of bound states in partial wave l



- δ-waves become repulsive at high momentum ↔ short distances
  ⇒ short-range repulsion of NN interaction

- Consider ^3P waves

  i) Central component ~ average of $^3P_0$, $^3P_1$ and $^3P_2$ is small at small energies
  ⇒ there must be something else ⇒ spin-orbit

  ii) Consider spin-orbit operator

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$= \frac{1}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]$$

$$\underset{\text{for } ^3P}{=} \frac{1}{2} J(J+1) - 2$$

$$= \begin{cases} -2 & \text{for } ^3P_0 \Rightarrow \text{repulsive} \\ -1 & \text{for } ^3P_1 \Rightarrow \text{repulsive} \longleftrightarrow \delta < 0 \\ +1 & \text{for } ^3P_2 \Rightarrow \text{attractive} \longleftrightarrow \delta > 0 \end{cases}$$

?? ⇒ there must be something else

tensor term

$V_{LS}\, \vec{L}\cdot\vec{S}$ where $V_{LS} < 0$

Experimental constraints : the deuteron

The deuteron is the only bound system of two nucleons.

Main properties

- Binding energy $E = 2.2245$ MeV
- Total angular momentum $J = 1$
- Magnetic dipole moment $\mu = 0.8574\, \mu_N$ with $\mu_N = \frac{e}{2M_p}$
- Electric quadrupole moment $Q = 0.2859$ e fm^2
- R.m.s. radius $r = 1.97$ fm

Proton-neutron system can exist in $T=0$ or $T=1$. Given $J=1$, there are four possibilities $^3S_1$, $^1P_1$, $^3P_1$, $^3D_1$

$$\begin{matrix} + & - & - & + \end{matrix} \longleftarrow \text{parity} = (-1)^L$$

Since parity has to be conserved, it can be at most a combination of the two states of the same parity.

Quadropole moment $= 0$ for a spherical nucleus

$\Rightarrow$ cannot be pure $L=0$ state

If it were a superposition of the P states, it would mix $T=0$ and $T=1$. Not impossible, but not very probable given that we expect isospin symmetry (specially since there is no Coulomb).

Further hints come from the magnetic moment.

$$\mu = \langle \Psi_{J\,M=J} \,|\, \mathcal{M}_z \,|\, \Psi_{J\,M=J} \rangle$$

for a nucleus in the state $|\Psi_{JM}\rangle$.

$\mathcal{M}_z$ is the third component of the magnetic moment operator

$$\vec{\mathcal{M}} = \mu_N \left[\sum_{i=1}^{Z} \left(\vec{l}_i + g_p \vec{s}_i \right) + \sum_{i=1}^{N} g_n \vec{s}_i \right]$$

proton orbital angular momenta

nucleon spin g-factors

$$g_p = 5.585 \qquad g_n = -3.826$$

For the deuteron one has

$$\mu = \mu_N \langle \Psi_{11} | \left(\vec{l}_p + g_p \vec{s}_p + g_n \vec{s}_n \right) | \Psi_{11} \rangle$$

and one finds

- for a superposition $|\Psi_{11}\rangle = \cos\omega_0 \, |{}^3S_1\rangle + \sin\omega_0 \, |{}^3D_1\rangle$

$$\mu = \mu_N \left(0.880 - 0.570 \sin^2\omega_0 \right)$$

- for a superposition $|\Psi_{11}\rangle = \cos\omega_1 \, |{}^1P_1\rangle + \sin\omega_1 \, |{}^3P_1\rangle$

$$\mu = \mu_N \left(0.500 + 0.190 \sin^2\omega_1 \right)$$

$\Rightarrow$ it has to be ${}^3S_1 + {}^3D_1$ (cf. experimental value $\mu = 0.8574 \, \mu_N$

The only term mixing different L components is the tensor
in the nuclear interaction

only operator that does not commute with $\vec{L}^2$