Nuclear reaction cross-sections and thermonuclear reaction rates

Nicolas de Séréville (nicolas.de-sereville@ijclab.in2p3.fr)
Laboratoire de Physique des 2 Infinis Irène Joliot Curie
Université Paris Saclay

Outline

Lecture 1: Introduction to nuclear astrophysics

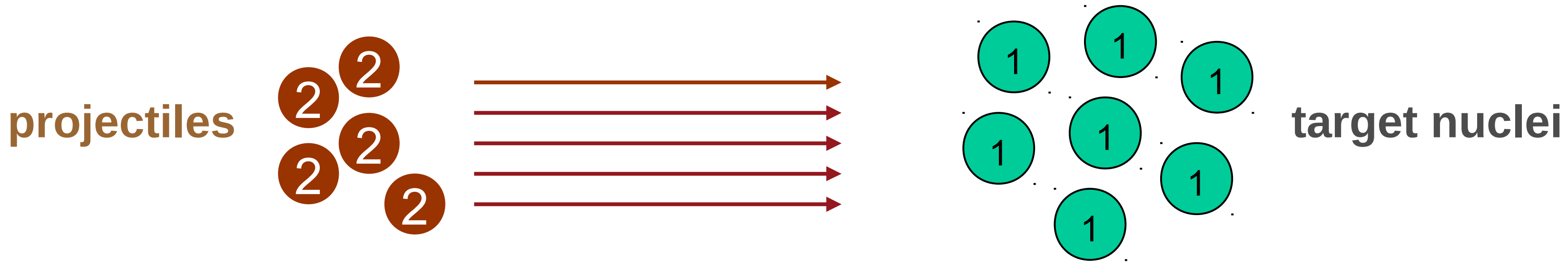
Lecture 2: Nucleosynthesis processes in the Universe

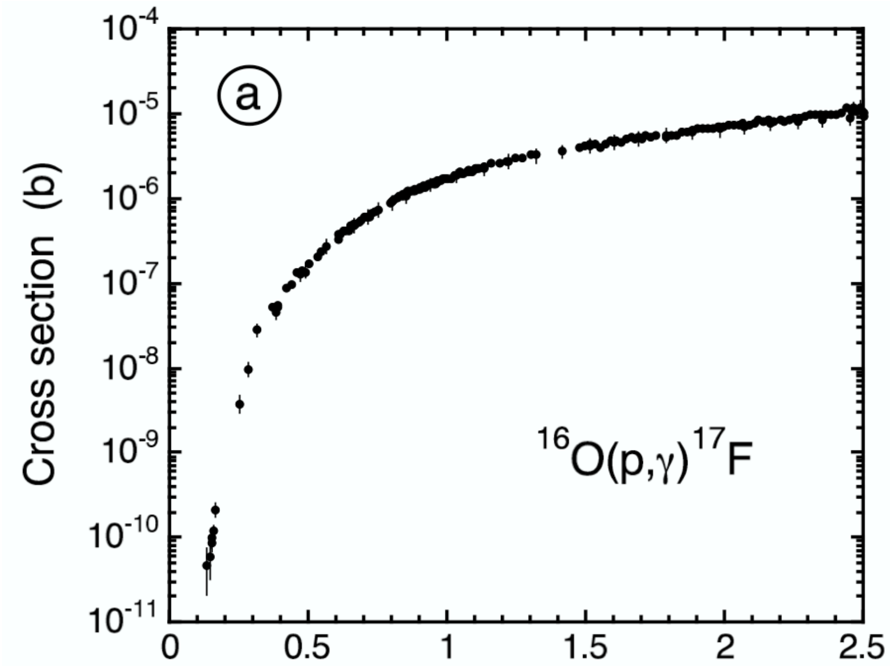
Lecture 3: Cross-sections and thermonuclear reaction rates

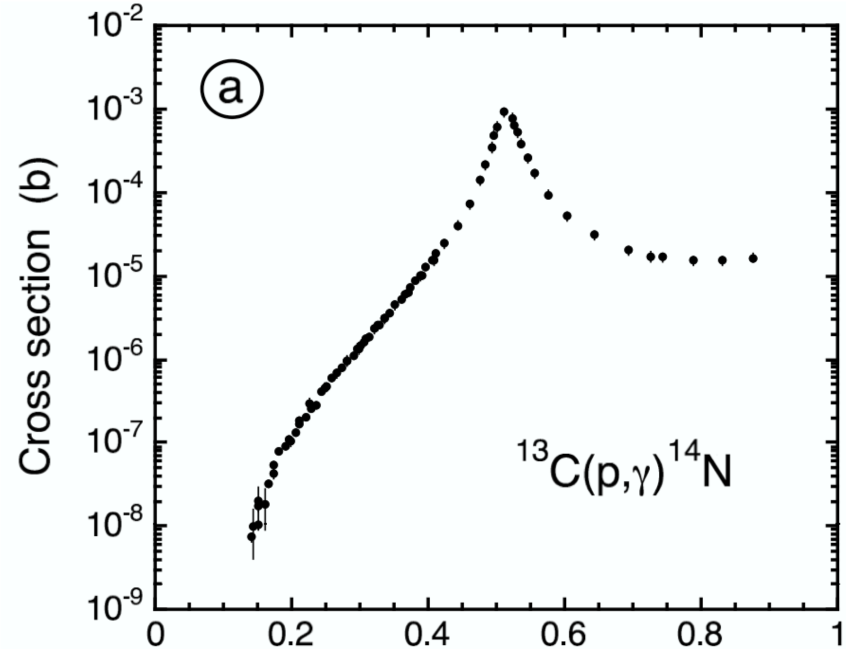
1. Nuclear reaction cross sections
   1. Definitions
   2. Quantum tunneling
   3. Astrophysical *S*-factor
   4. Reaction mechanisms, Breit-Wigner cross-section...
2. Thermonuclear reaction rates
   1. Definitions
   2. Gamow window
   3. Non-resonant & resonant reaction rates for charged particules
3. Additional effects on thermonuclear reaction rates

Lecture 4: Experimental approaches in nuclear astrophysics

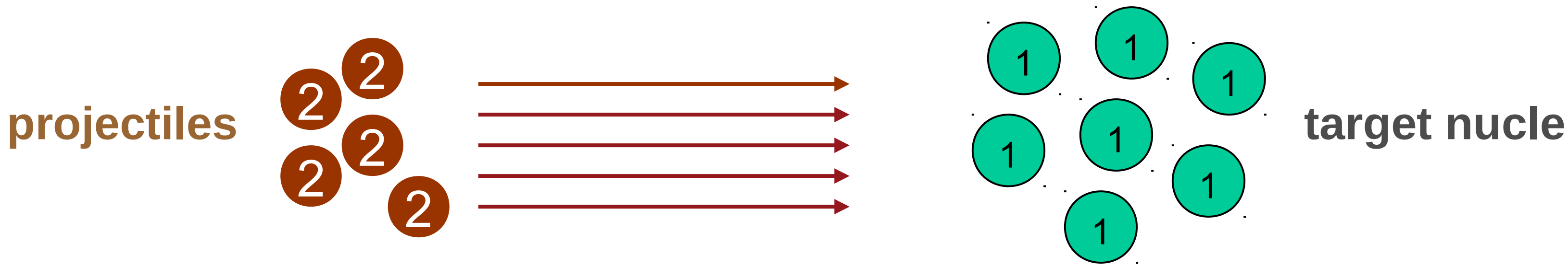
1. Reaction cross-section







Definition of cross-section



- Cross-section of the reaction 1 + 2 → 3 + 4 [notation 1(2,3)4] is defined as:

$$\frac{\text{number of reactions / second}}{(\text{nb of projectiles / cm}^2\text{ / second})\ (\text{nb of target nuclei within the beam})}$$

= surface presented by 1 to the projectile 2 for a given reaction

- "Billiard-type" description of the cross-section

  $\sigma = \pi(R_1 + R_2)^2$ with the nuclear radius $R_N \approx 1.3\ A^{1/3}$ fm (10^{-13} cm)

→ $\sigma(^1H + {}^1H) = 0.2\times10^{-24}\ cm^2$ $\sigma(^{238}U + {}^{238}U) = 8.2\times10^{-24}\ cm^2$

→ unit of nuclear cross-sections: 1 barn (b) = 10^{-24} cm^2

The maximum reaction cross-section

- A nuclear reaction is any process which is different from elastic scattering → fraction of incoming particles change identity or kinetic energy

- Quantum description of the maximum reaction cross-section

$$\sigma_{max} = (2l+1)\pi\lambda\!\!\!^{-2} \qquad \text{where} \qquad \lambda\!\!\!^{-} = \frac{\hbar}{\sqrt{2\mu E}} = \frac{m_1 + m_2}{m_1}\frac{\hbar}{\sqrt{2m_2E_2}}$$

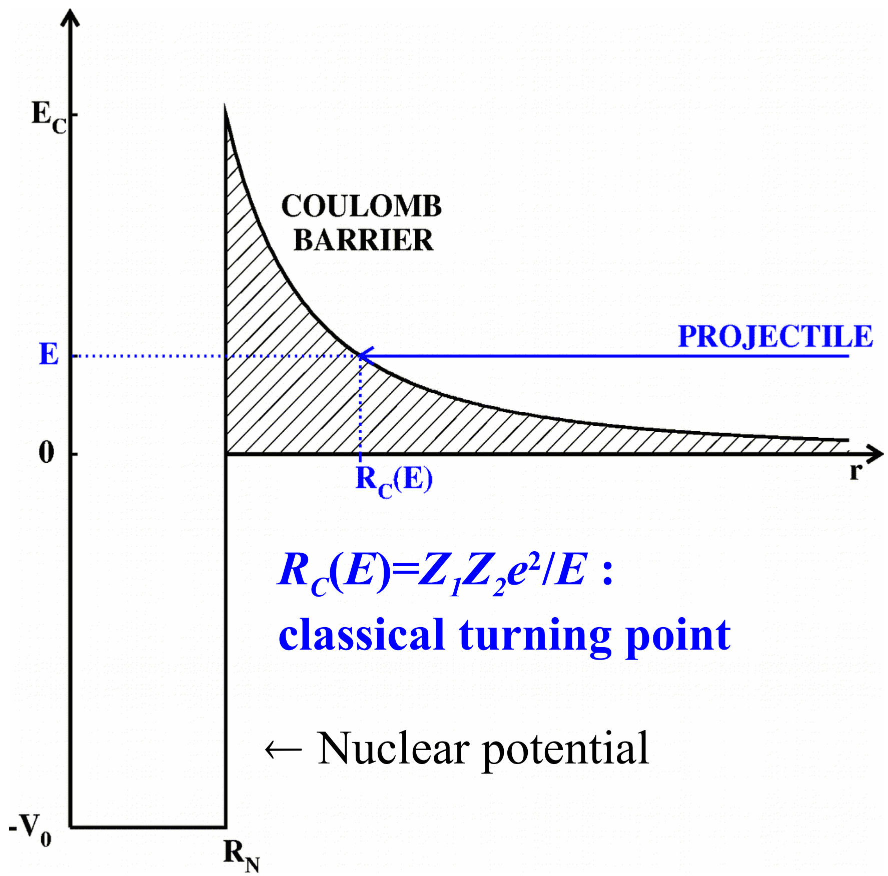
  is the de Broglie wavelength, E the total kinetic energy in the center-of-mass system of reference, and $\mu = m_1m_2 / (m_1+m_2)$ the reduced mass.

  Note that $\sigma_{max} \propto 1/E$

  The statistical factor $(2l+1)$ corresponds to the number of eigenstates of the system 1+2 of angular momentum $\boldsymbol{L}$ (l is the orbital quantum number)

- $\sigma < \sigma_{max}$ in part. because of the centrifugal and Coulomb barriers

The Coulomb and centrifugal barriers



- **Coulomb barrier:** in a reaction between charged particles (atomic numbers Z_1, Z_2)

$$V_{coul}(r) = \frac{Z_1 Z_2 e^2}{r} = 1.44 \frac{Z_1 Z_2}{r(\mathrm{fm})} (\mathrm{MeV})$$

- **Centrifugal barrier:** energy needed to move closer 1 and 2 to a distance r given the orbital momentum ***L***

$$V_{cent}(r) = \frac{\left\|\vec{L}\right\|^2}{2\mu r^2} \Longrightarrow V_{cent}(r) = \frac{l(l+1)\hbar^2}{2\mu r^2}$$

$l(l+1)\hbar^2$ eigenvalues of $\boldsymbol{L}^2$

Remarks:

1) in stars, $T_c \sim 10^7 - 10^9$ K
→ $kT_c \sim 1 - 100$ keV < $V_{coul}(R_N)$
e.g. V_{coul}(p+p) = 550 keV

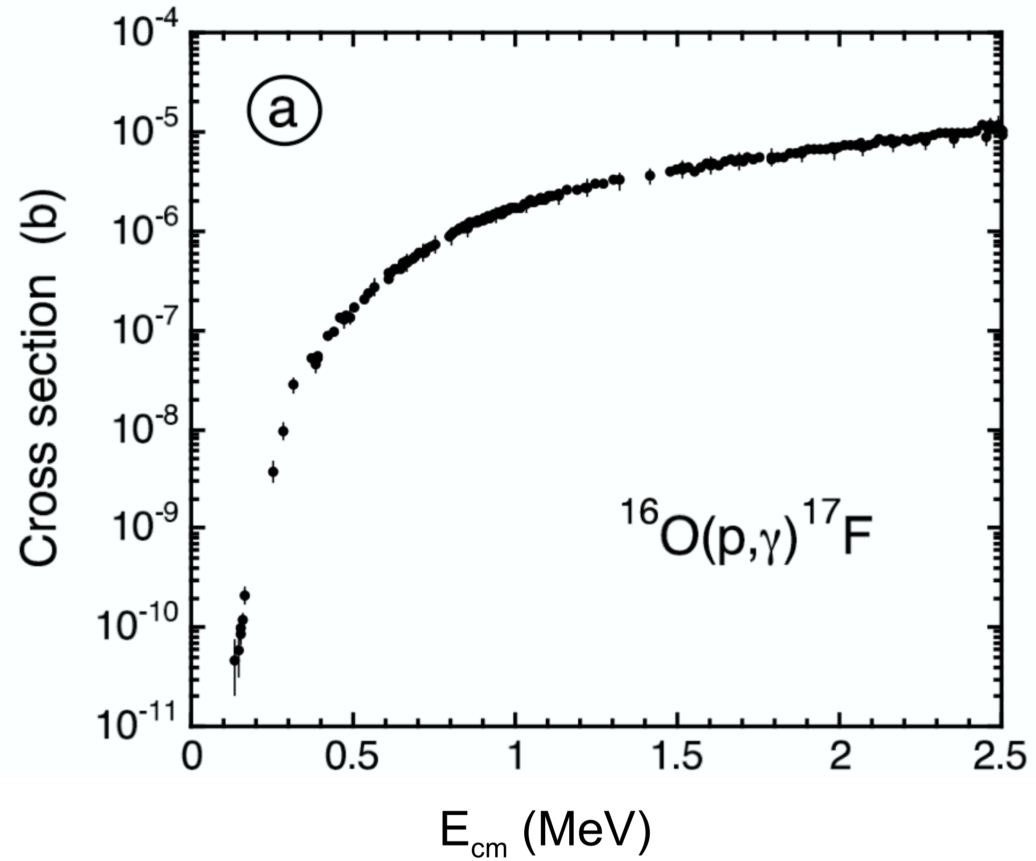
⇒ Penetration of the Coulomb barrier by the "tunnel effect"

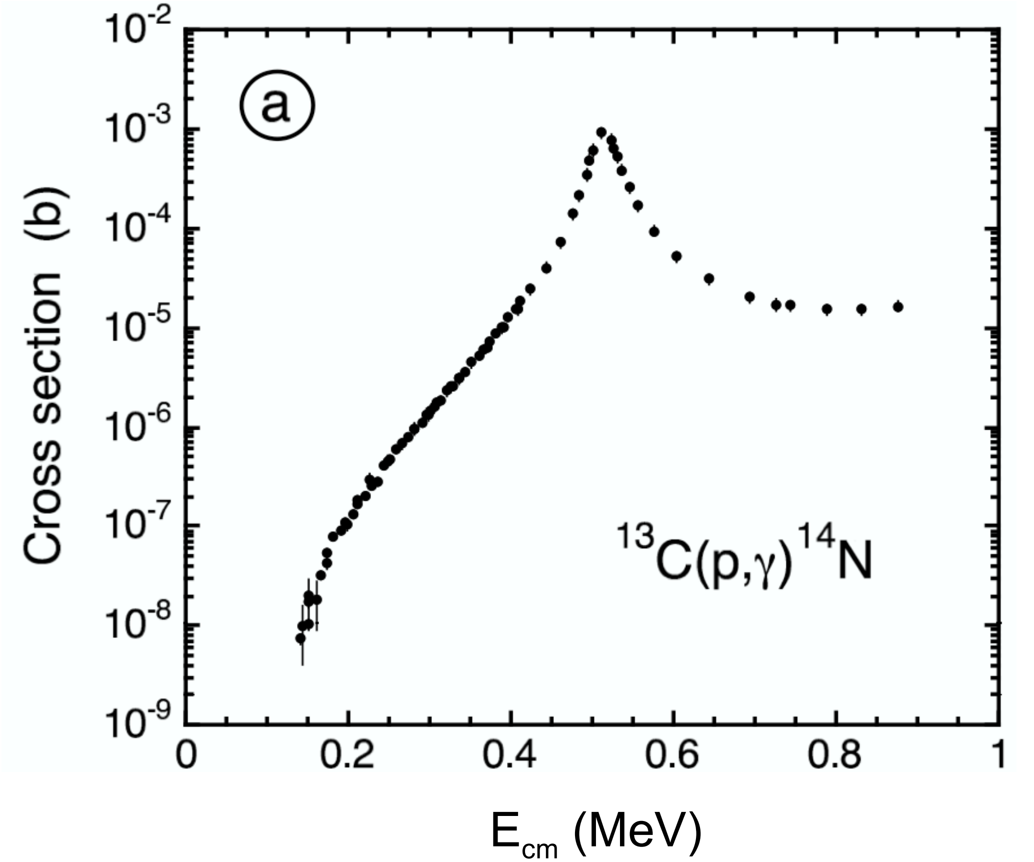
2) if $A_1+A_2 \sim A_1$, then:

$$\frac{V_{cent}(R_N)}{V_{coul}(R_N)} \approx \frac{10 \times l(l+1)}{A_2 \left(A_1^{1/3} + A_2^{1/3}\right) Z_1 Z_2}$$

⇒ cross sections between light nuclei are "negligible" for non-head-on collisions (ℓ ≠ 0)

Experimental cross sections

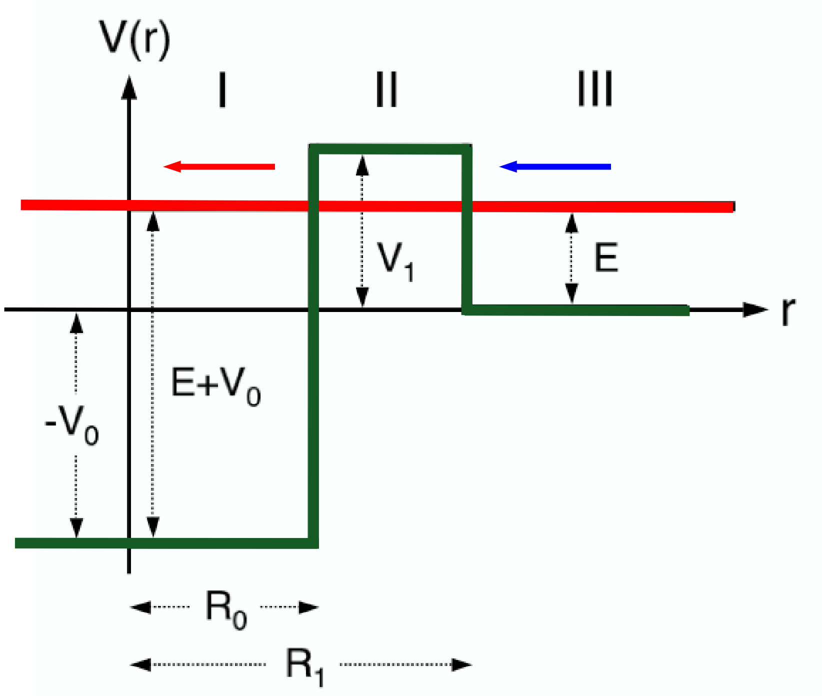




- Why does the cross-section fall drastically at low energies?
- What is the origin of the peak in the cross section?

The tunnel effect – 1D (1)

Square-barrier potential with ℓ = 0 →



The radial wave functions *u(r)* **(1D)** are solution of the time-independent Schrödinger equation

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r)\right]u = 0$$

Solutions

$$u_{III} = Fe^{ikr} + Ge^{-ikr} \quad \text{with} \quad k^2 = \frac{2m}{\hbar^2}E$$

$$u_{II} = Ce^{-\kappa r} + De^{\kappa r} \quad \text{with} \quad \kappa^2 = \frac{2m}{\hbar^2}(V_1 - E)$$

$$u_I = Ae^{iKr} + Be^{-iKr} \quad \text{with} \quad K^2 = \frac{2m}{\hbar^2}(E + V_0)$$

Continuity conditions

$$(u_I)_{R_0} = (u_{II})_{R_0}$$

$$\left(\frac{du_I}{dr}\right)_{R_0} = \left(\frac{du_{II}}{dr}\right)_{R_0}$$

$$(u_{II})_{R_1} = (u_{III})_{R_1}$$

$$\left(\frac{du_{II}}{dr}\right)_{R_1} = \left(\frac{du_{III}}{dr}\right)_{R_1}$$

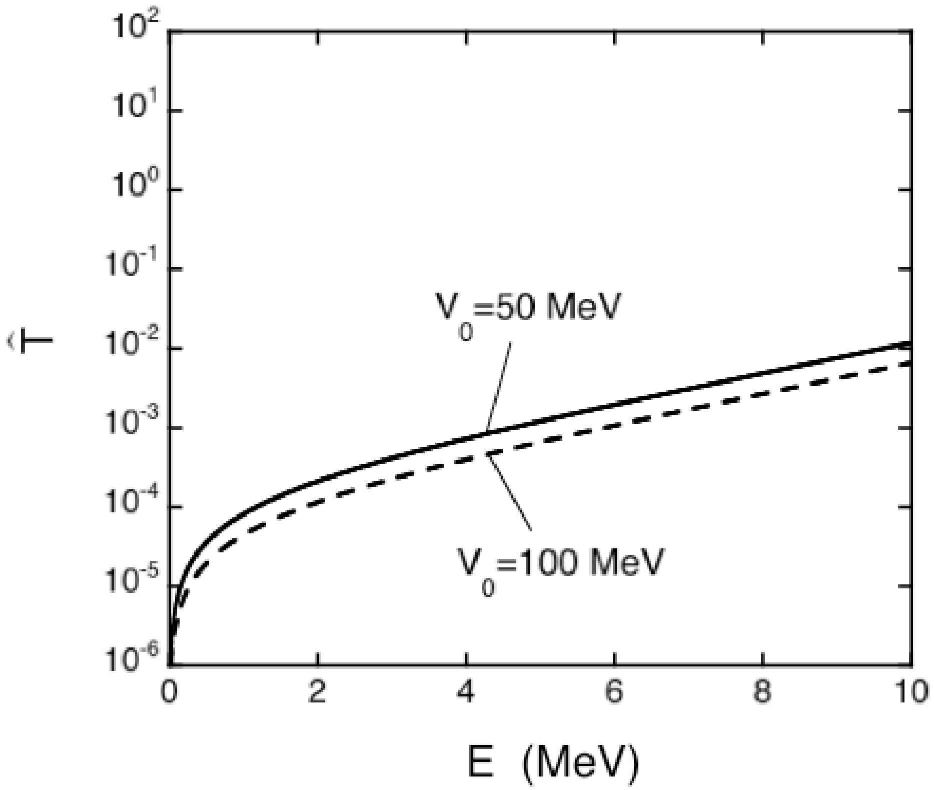
Transmission coefficient

- Ratio of transmitted to incident current densities (of fluxes), e.g. $j_{inc} = v_{III}\left|G\right|^2 = \hbar k/m\left|G\right|^2$
- $$\widehat{T} = \frac{K}{k}\frac{\left|B\right|^2}{\left|G\right|^2} \approx e^{-(2/\hbar)\sqrt{2m(V_1 - E)}(R_1 - R_0)}$$

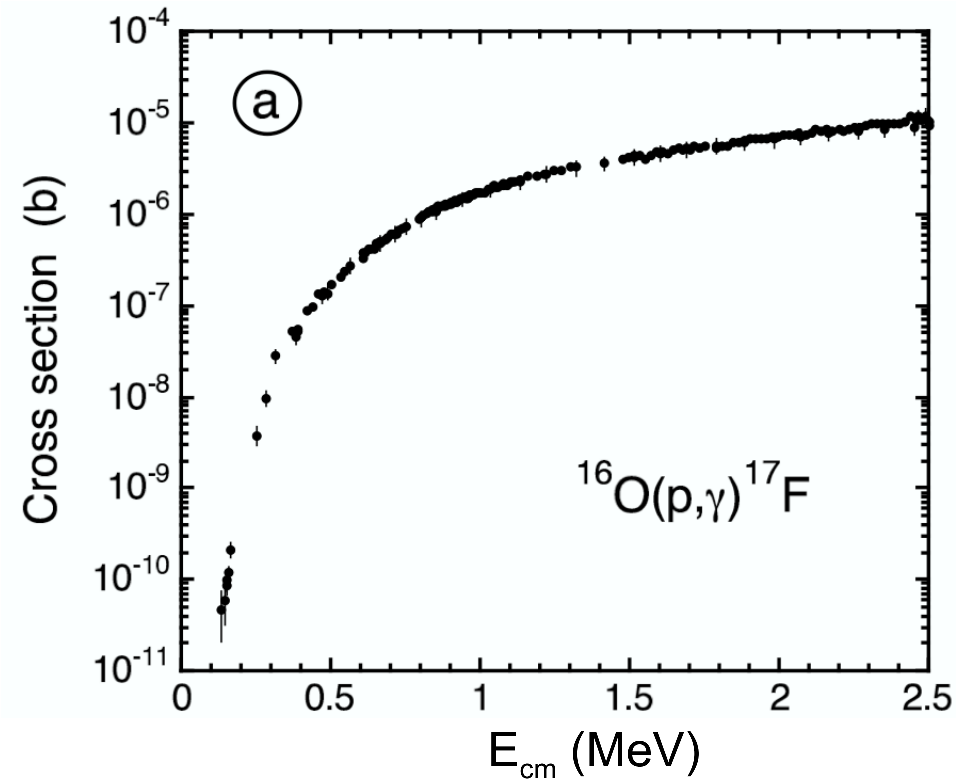
Limit of low *E*

The tunnel effect – 1D (2)

Calculated



Experimental



The tunnel effect is the reason for the strong drop in cross-section at low energies!

The tunnel effect – 3D

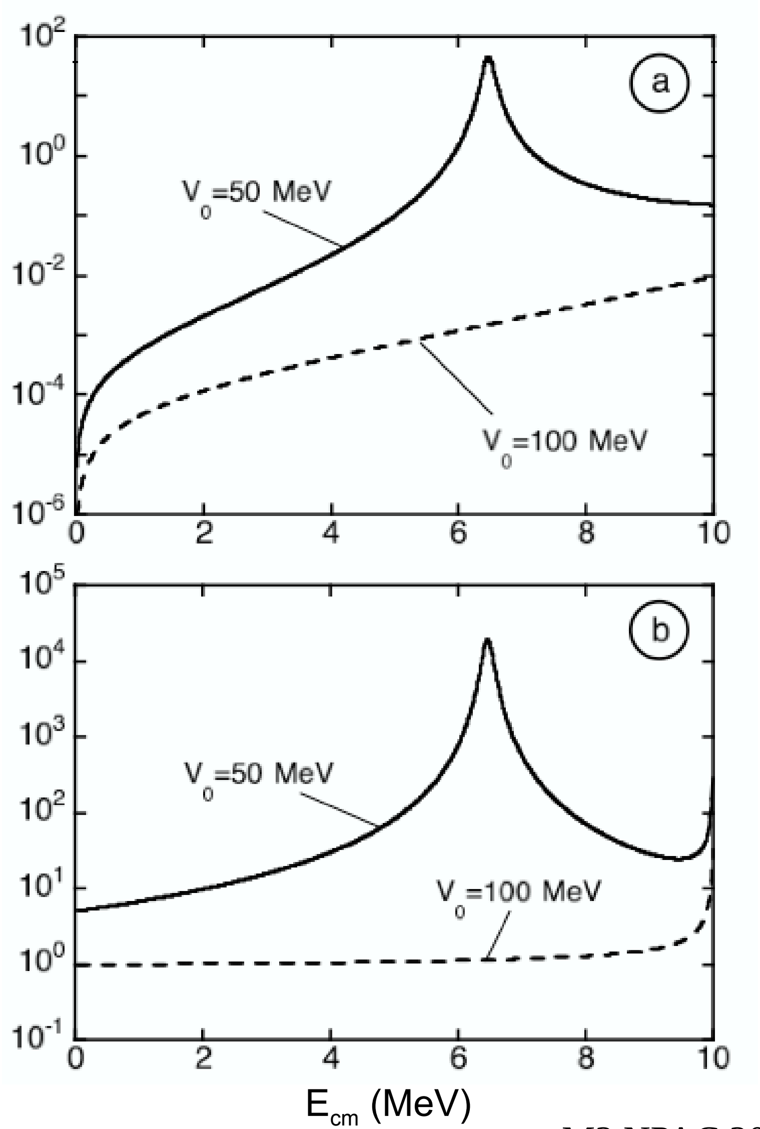
Radial wave functions for a 3D “square”-barrier potential:

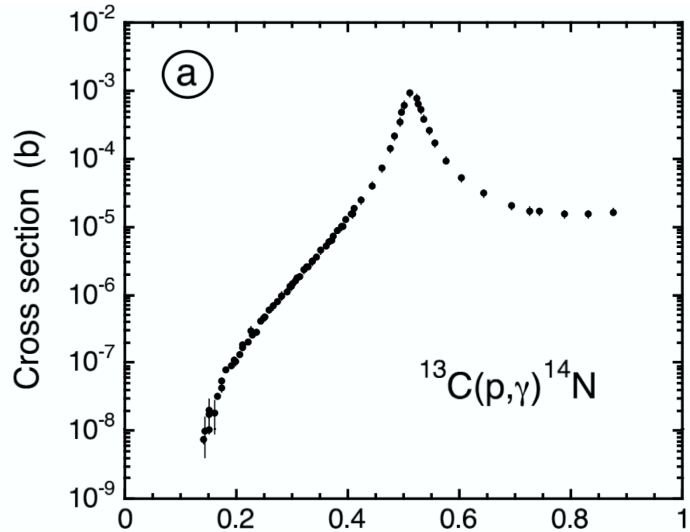
- Same continuity conditions
- Emergence of resonance phenomenon

$$u_{III} = \boxed{F'} \sin(kr + \delta_0)$$
$$u_{II} = Ce^{-\kappa r} + De^{\kappa r}$$
$$u_I = \boxed{A'} \sin(Kr)$$

$$\frac{|A'|^2}{|F'|^2}$$

$$\frac{|A'|^2}{|F'|^2}\frac{1}{\hat{T}}$$

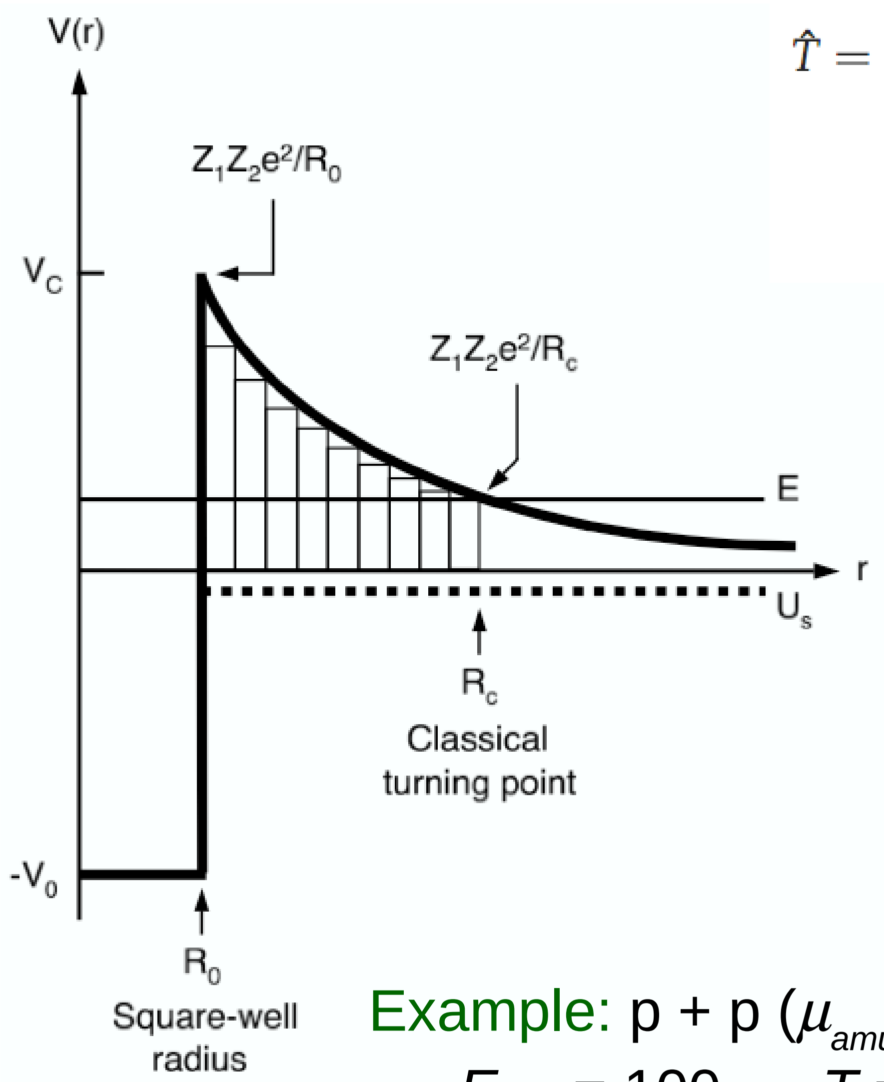




A resonance results from favorable wave function matching conditions at the boundaries

Different V_0 values mean different wavelength in the interior region.

Transmission through the Coulomb barrier



$$\hat{T} = \hat{T}_1 \cdot \hat{T}_2 \cdot \ldots \cdot \hat{T}_n \approx \exp\left[-\frac{2}{\hbar}\sum_i \sqrt{2m(V_i - E)}(R_{i+1} - R_i)\right]$$

$$\xrightarrow[n \text{ large}]{} \exp\left[-\frac{2}{\hbar}\int_{R_0}^{R_c} \sqrt{2m[V(r) - E]}\, dr\right]$$

$$\widehat{T} \approx \exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{2E}} Z_1 Z_2 e^2\right) = \exp\left(-2\pi\eta\right)$$

(Zero angular momentum)

- η: Sommerfeld parameter
- exp(-2πη): Gamow factor

$$\rightarrow \; 2\pi\eta = 31.29 Z_1 Z_2 \sqrt{\frac{\mu_{\text{amu}}}{E_{\text{keV}}}}$$

Example: p + p (μ_{amu} = 1/2)

- E_{keV} = 100 → T = 11%
- E_{keV} = 6 → T = 0.01% (in Sun)

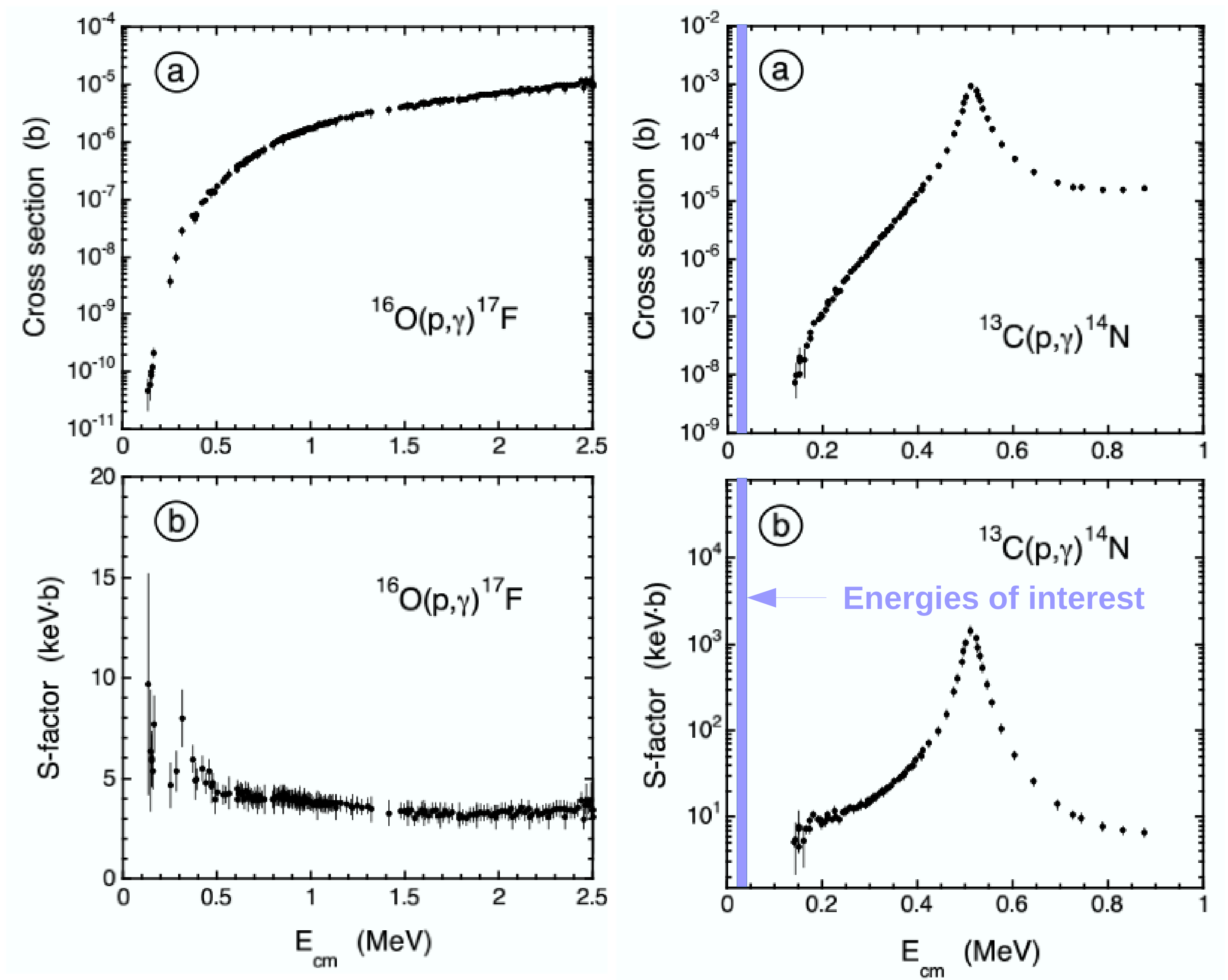
The astrophysical S-factor

$$\sigma(E) \equiv \boxed{\frac{1}{E}} \times \boxed{e^{-2\pi\eta}} \times \boxed{S(E)}$$

Correction of the effect $\sigma_{max} \propto 1/E$

Correction of the tunneling probability ($\ell = 0$)

S(E): astrophysical *S*-factor which contains all the nuclear effects for a given reaction

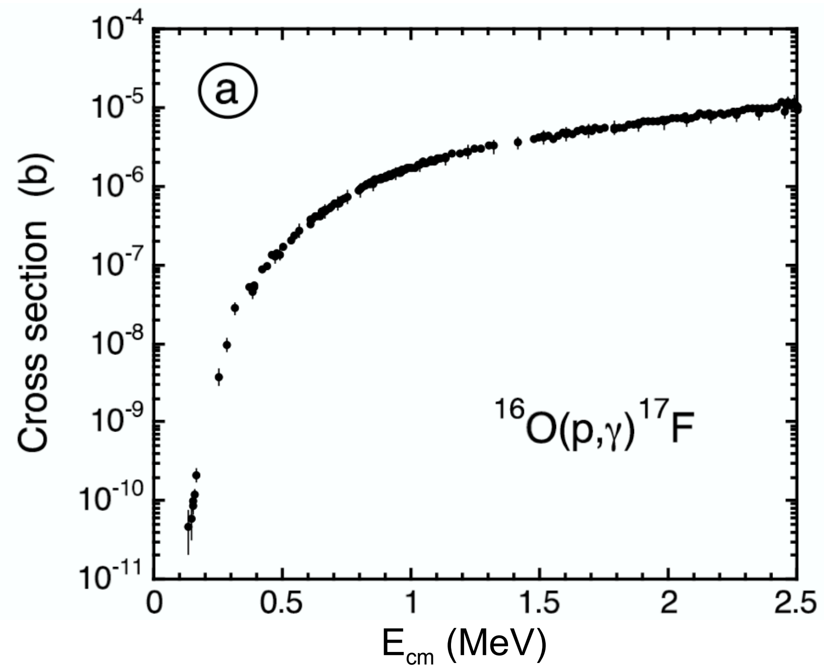


- (sometimes) S(E) is a smoothly varying function
- Most of the cases, extrapolation to astrophysical energies needed!

Direct and resonant captures

Let's consider the reaction A(a,b)B where b can be a particle or a photon

Direct capture

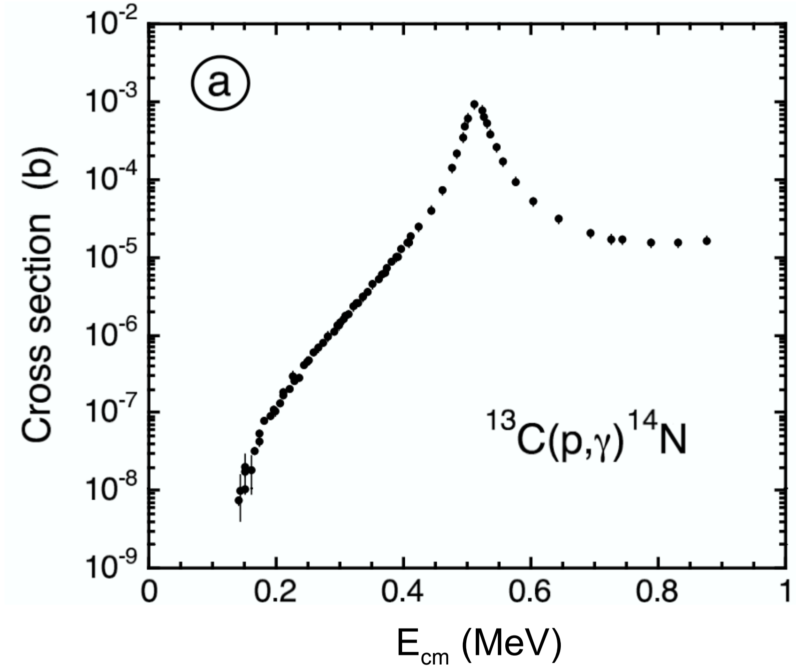


- One step process leading to final nucleus B
- Single matrix element

$$\sigma \propto |\langle b + B\,|H|\,a + A\rangle|^2$$

- Occurs at all interaction energies
- Weak energy dependence of *S*-factor

Resonant capture



- Two steps process

  1) Formation of compound nucleus a + A → C*

  2) Decay of compound nucleus C* → b + B

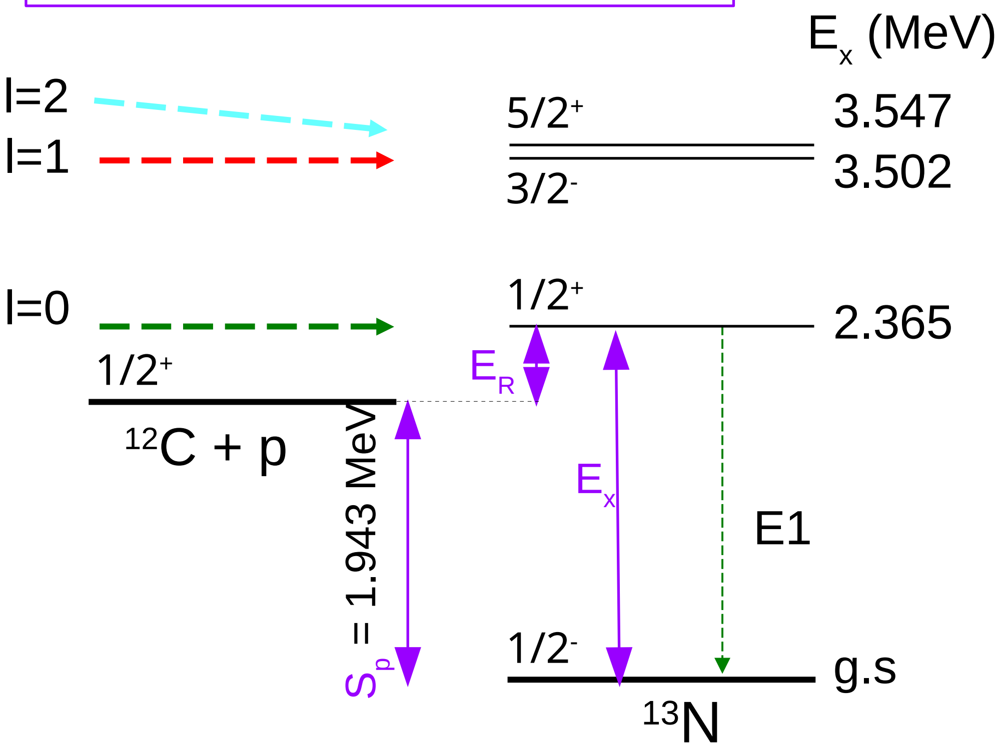
- Product of two matrix elements

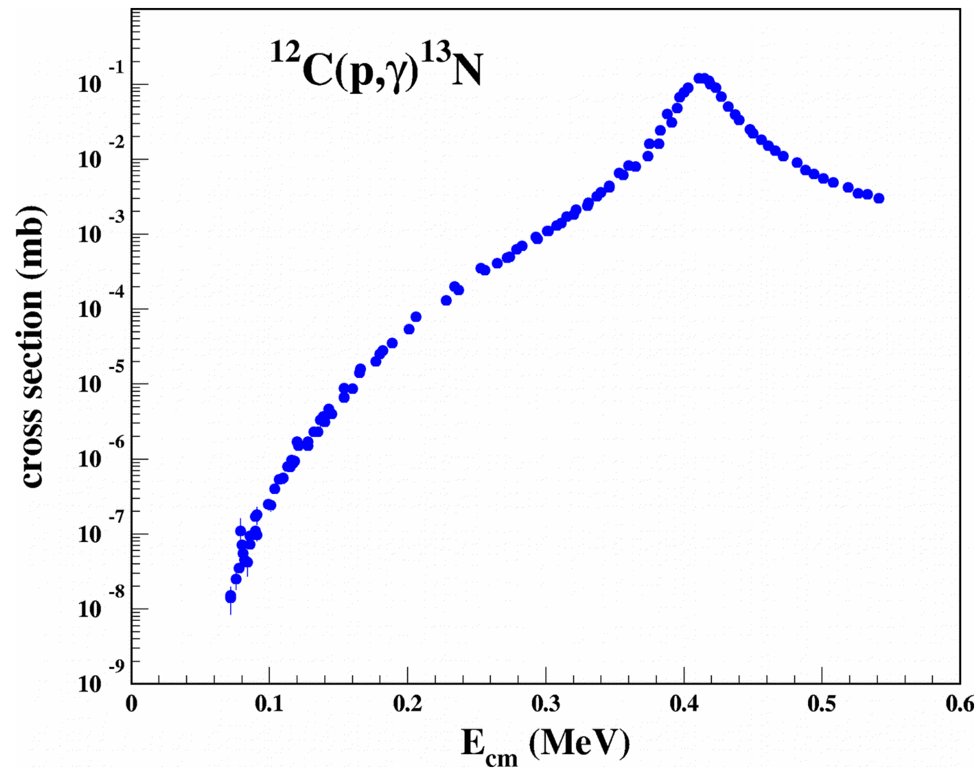
$$\sigma \propto |\langle b + B\,|H_1|\,C^*\rangle|^2 \times |\langle C^*\,|H_2|\,a + A\rangle|^2$$

- Occurs at specific energies
- Strong energy dependence of *S*-factor

Resonant capture

A simple case: $^{12}C(p,\gamma)^{13}N$





- Reaction *Q*-value (Δ = mass excess)
  - → $Q = \Delta(^{12}C) + \Delta(p) - \Delta(^{13}N) = 1.943$ MeV
- Particle (proton) separation energy S_p
  - → $S_p = \Delta(^{12}C) + \Delta(p) - \Delta(^{13}N)$
- Resonance energy (in center of mass)
  - → $E_R = E_x - S_p = 2.365 - 1.943 = 422$ keV

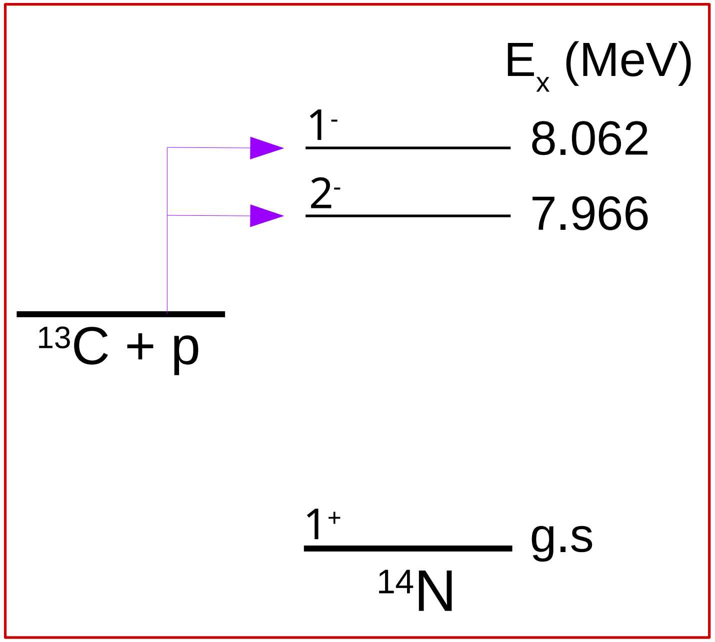
- Relative angular momentum ℓ
  - $\mathbf{J}_R = \mathbf{J}(^{12}C) + \mathbf{J}(p) + \ell = 1/2$
  - $\pi_R = \pi(^{12}C).\pi(p).(-1)^{\ell} = +1$
  - → $\ell = 0$
- Coupling scheme (start with entrance channel)

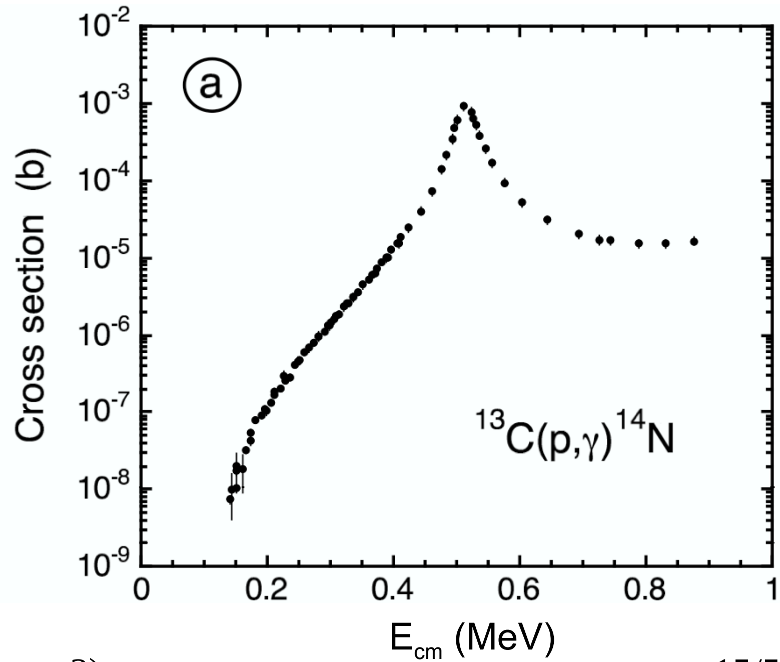
Resonant capture: your turn!

The first two resonant states in the $^{13}C(p,\gamma)^{14}N$ reaction have known energy and spin/parity.

1) Calculate the Q-value of the reaction, and determine the resonance energies. Compare with experimental data.
2) Calculate the relative orbital angular momentum ℓ needed to form these states

Useful information: $J^{\pi}(^{13}C) = 1/2^-$, $\Delta(^{13}C) = 3.125$ MeV, $\Delta(^{1}H) = 7.289$ MeV, $\Delta(^{14}N) = 2.863$ MeV.





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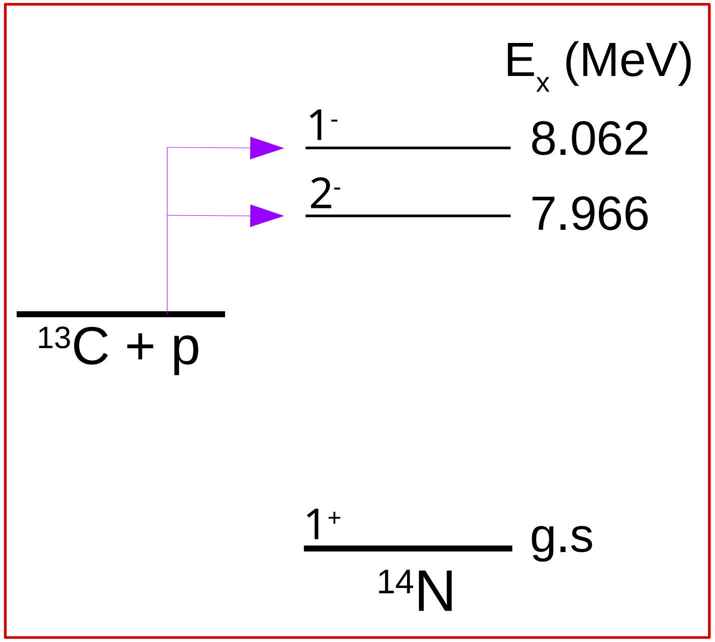

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1) $S_p = Q = \Delta(^{13}C) + \Delta(^1H) - \Delta(^{14}N) = 7.551$ MeV

$E_R(2-) = 7.966 - 7.551 = 415$ keV

$E_R(1-) = 8.062 - 7.551 = 511$ keV

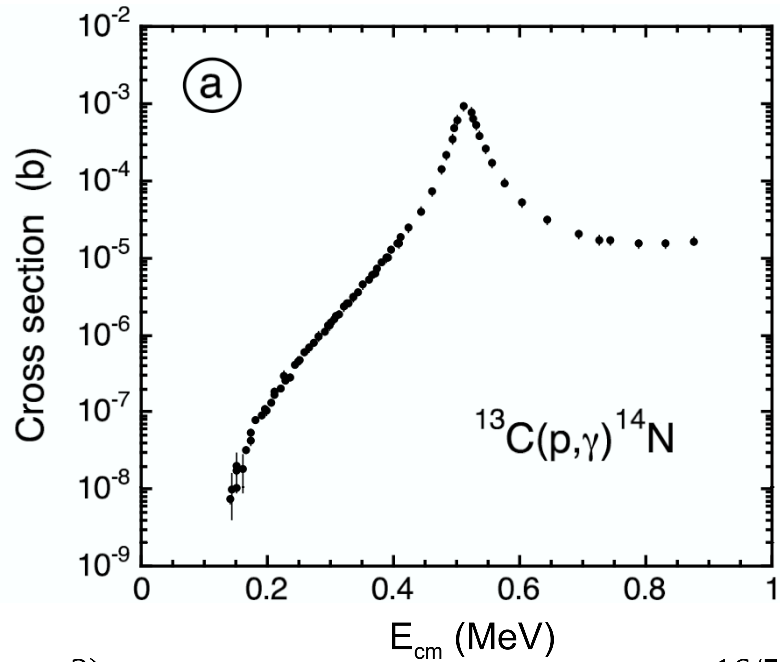
2) Entrance channel:

- Channel spin: $|J(^{13}C)-J(p)| \leq s \leq |J(^{13}C)+J(p)|$ $\rightarrow s = 0, 1$
- Parity: $\pi = \pi(^{13}C).\pi(p) = -1 \times +1 = -1$

Resonances:

- Negative parity states: $\pi_R = \pi.(-1)^\ell \rightarrow \ell$ even
- $\mathbf{J_R} = \boldsymbol{s} + \boldsymbol{\ell} \rightarrow |s-\ell| \leq J_R \leq |s+\ell|$

  $\rightarrow \ell = 0 \Rightarrow J_R = 0, 1$; $\ell = 2 \Rightarrow J_R = 1, 2, 3$



Direct capture: your turn!

Explain why the $^{16}O(p,\gamma)^{17}F$ reaction proceeds through direct capture and not resonant capture

$^{17}_{9}F_{8}$-1 From ENSDF - Evaluated December 1992 $^{17}_{9}F_{8}$-1

Adopted Levels, Gammas 1993Ti07

History

Type	Author	Citation	Literature Cutoff Da
Full Evaluation	J. H. Kelley, D. R. Tilley, H. R. Weller and C. M. Cheves	NP A564, 1 (1993)	31-Dec-1992

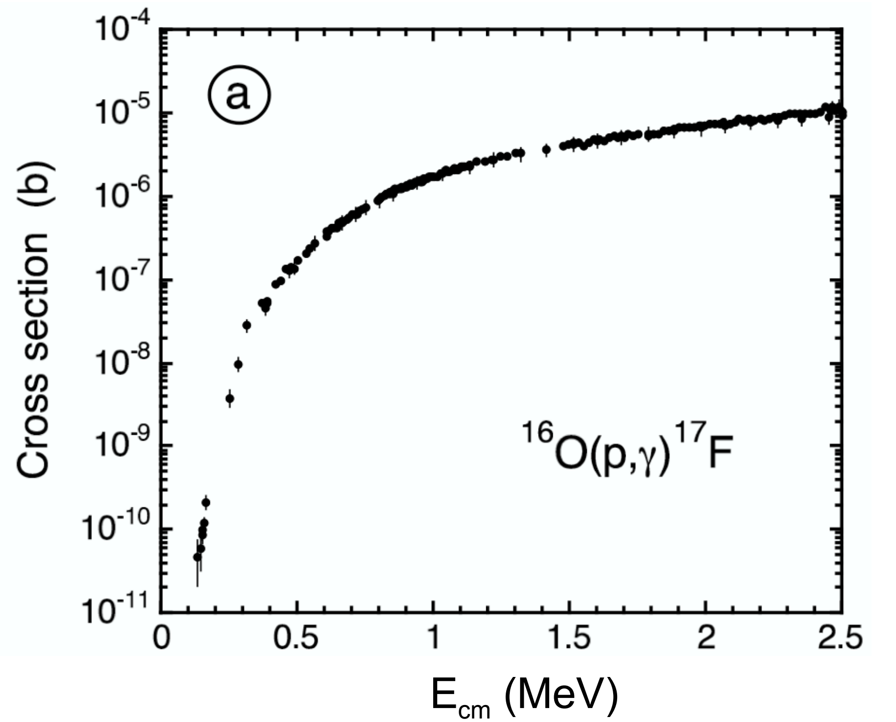
$Q(\beta^-)$=−14548.7 *5*; S(n)=16800 *9*; S(p)=600.27 *25*; $Q(\alpha)$=−5818.7 *4* 2012Wa38
Note: Current evaluation has used the following Q record.
$Q(\beta^-)$=−14538 *50*; S(n)=16800 *8*; S(p)=600.27 *23*; $Q(\alpha)$=−5818.67 *37* 1997Au04
See other reaction references in 1993Ti07.

^{17}F Levels

Cross Reference (XREF) Flags

A	^{17}F β^+ decay	E	$^{16}O(p,\gamma)$	I	$^{17}O(p,n)$
B	$^{14}N(^3He,\gamma)$	F	$^{16}O(p,p)$,$^{16}O(p,\alpha)$	J	^{17}Ne β^+ decay
C	$^{14}N(^6Li,t)$	G	$^{16}O(d,n)$		
D	$^{15}N(^3He,n)$	H	$^{16}O(^3He,d)$		

E(level)	J^π	$T_{1/2}$	XREF	Comments
0.0	$5/2^+$	64.49 s *16*	ABCDE GHIJ	$\%\varepsilon+\%\beta^+$=100 T=1/2; μ=+4.7223 *12* (1989Ra17) $T_{1/2}$: weighted average of : 64.31 s *9* (1977Az01), 64.50 s *25* (1972Al42) 65.2 s *2* (1969Wo09).
495.33 *10*	$1/2^+$	286 ps *6*	BCDE GHIJ	
3104 *3*	$1/2^-$	19 keV *1*	BCDEFGH J	%IT=6.3×10^{-5} *11*; %p=100 Γ_γ=0.012 eV *2*
3857 *4*	$5/2^-$	1.5 keV *2*	BCDEFGH	%IT=0.0073 *17*; %p=100 Γ_γ=0.11 eV *2*



https://www.nndc.bnl.gov/ensdf/

Nuclear resonance profile

Energy profile of excited nuclear states

- Time-dependent wave function:

$$\Psi(t) = \Psi(0)\, e^{-\frac{i}{\hbar}E_R t} \times e^{-\frac{t}{2\tau}}$$

where τ is the mean lifetime of the excited state

- The wave function as a function of energy is obtained by the Fourier transform (conjugate variables):

$$\phi(E) = \int_0^{\infty} \Psi(t)\, e^{\frac{i}{\hbar}Et} dt$$

- The probability distribution is then:

$$f_R(E) = |\phi(E)|^2 = \frac{\hbar}{2\pi\tau} \frac{1}{(E - E_R)^2 + (\hbar/2\tau)^2}$$

= Breit-Wigner profile (Cauchy-Lorentz distribution)

Full width at half maximum $\Gamma = \frac{\hbar}{\tau}$ ⇐ Heinsenberg uncertainty principle

Particle partial width

- Partial width (energy unit): $\Gamma_a = \hbar \lambda_a$ where λ_a is the probability per unit time that the "decay" particle *a* (p, n, α, …) passes through a large spherical surface at a distance r, r → ∞:

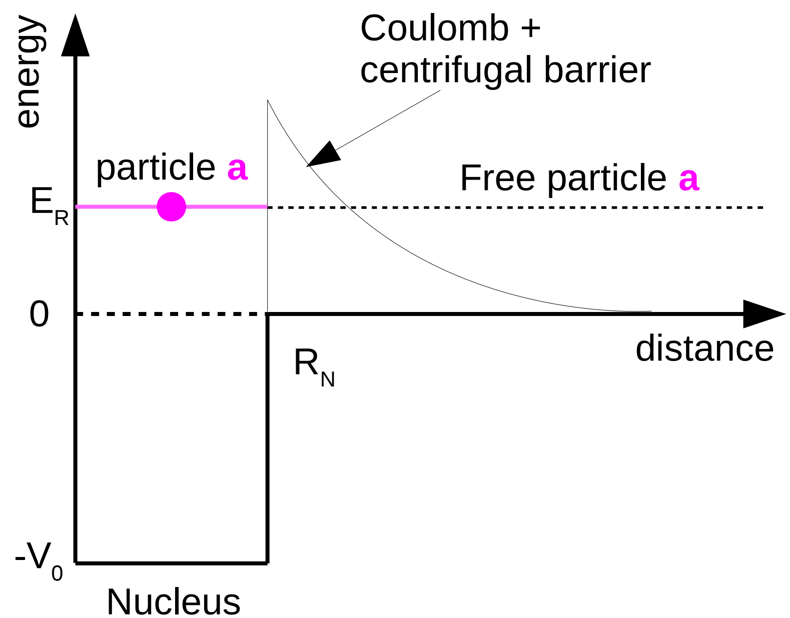
$$\lambda_a = \lim_{r\to\infty} v \iint_{d\Omega} |\Psi(r,\theta,\phi)|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$\lambda_a = \lim_{r\to\infty} v \iint_{d\Omega} \left|\frac{u(r)}{r}\right|^2 |Y_{lm}(\theta,\phi)|^2 r^2 \sin\theta \, d\theta \, d\phi = v\,|u_l(\infty)|^2$$

v being the relative velocity, and $Y_{lm}(\theta,\phi)$ the spherical harmonics

- With the penetration factor for the Coulomb and centrifugal barriers

$$P_l(E, R_N) = \frac{|u_l(\infty)|^2}{|u_l(R_N)|^2} \Rightarrow \boxed{\Gamma_a = \hbar \sqrt{\frac{2E}{\mu}} P_l(E, R_N)\,|u_l(R_N)|^2}$$



- The partial width is the product of two factors:
  - Probability of appearance of particle *a* at the nuclear radius R_N
  - Probability that particle *a* pass through Coulomb and centrifugal barrier

Gamma-ray transitions

- Multipole expansion of the electromagnetic operator: Q_L^{EM}

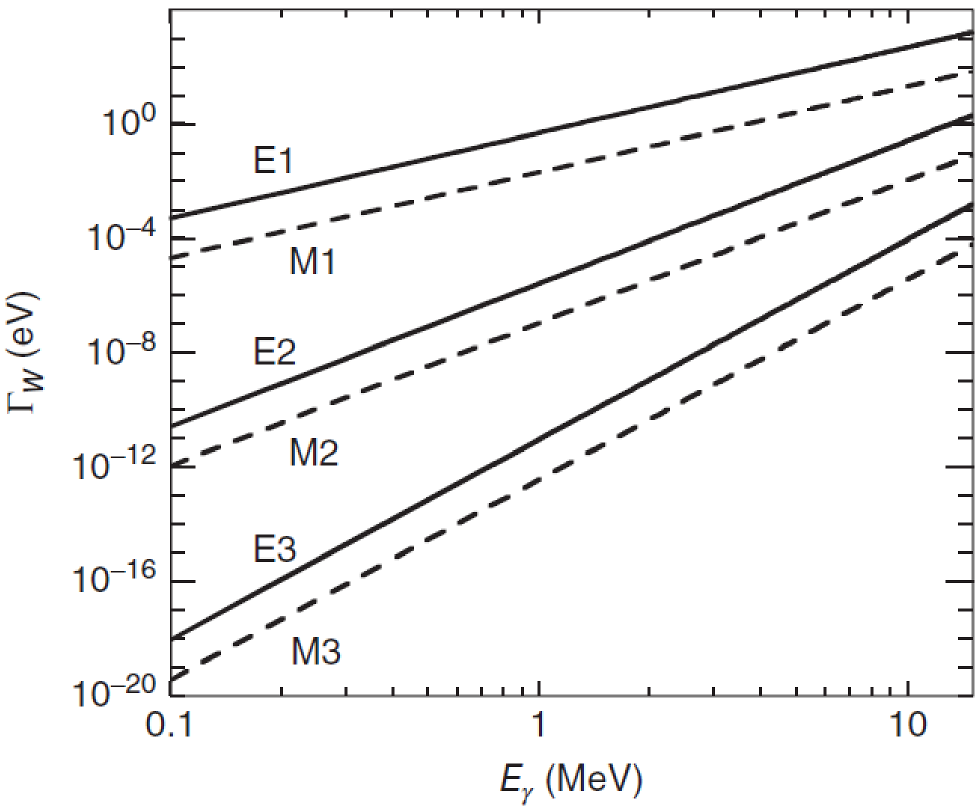
  Transition rate $\Rightarrow \lambda_L \propto \left\langle \Psi_f \left| Q_L^{EM} \right| \Psi_i \right\rangle^2$

- Selection rules (conservations of angular momentum and parity):

$$|J_i - J_f| \leq L \leq |J_i + J_f|$$
$$\pi_i = \pi_f \, (-1)^L \qquad \text{if electric}$$
$$\pi_i = \pi_f \, (-1)^{L+1} \qquad \text{if magnetic}$$

- Weisskopf estimate = jump of a proton from one shell-model state to another, assuming the nucleus consists of an inert core plus a proton

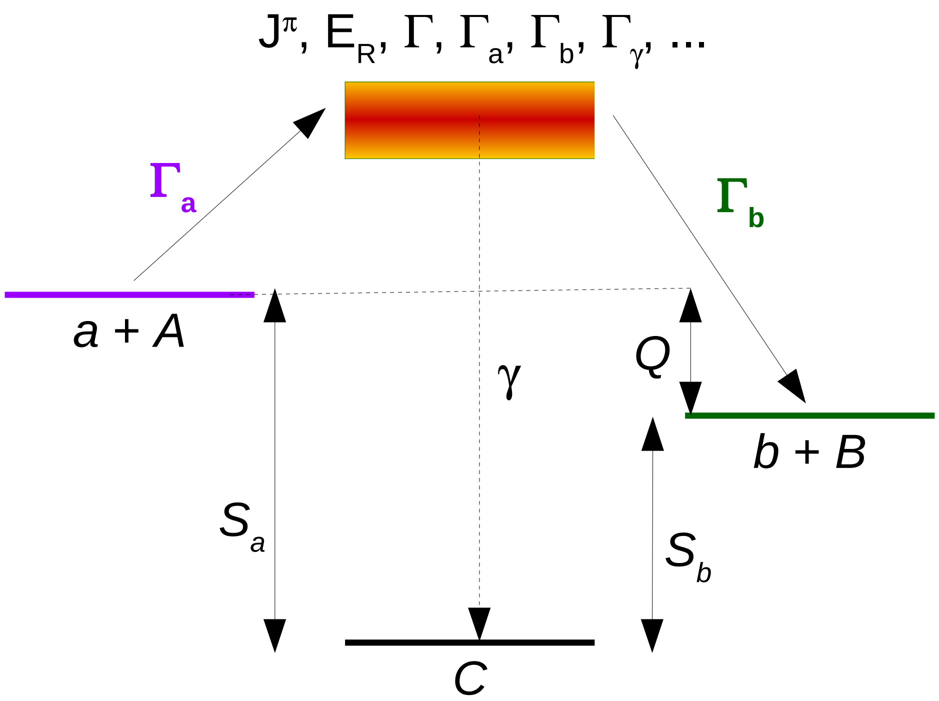
$$\Rightarrow \Gamma_\gamma^L = \hbar\lambda_L = \alpha_L^{EM} \, E_\gamma^{2L+1}$$



Resonant capture

Let's consider the $a + A$ reaction proceeding through the formation of compound nucleus C^*

$$a + A \rightarrow C^* \rightarrow b + B$$
$$\hookrightarrow \gamma + C$$



Q-value, particle emission threshold S_a(C), S_b(C), and resonance energy

- Q-value for A(a,b)B $\rightarrow Q = \Sigma\Delta_i - \Sigma\Delta_f$
- $S_a = \Delta(a) + \Delta(A) - \Delta(C)$
- $E_R = E_x - S_a$ (Note: the resonance energy depends on the channel!)

Partial and total widths

- Γ_a: formation probability of the compound nucleus C^* from the $a+A$ entrance channel
- Γ_b: decay probability of the compound nucleus C^* to the $b+B$ exit channel
- Γ_γ: γ-ray decay probability of the compound nucleus C^* to its ground-state
- $\Gamma = \Gamma_a + \Gamma_b + \Gamma_\gamma + \ldots$

The Breit-Wigner cross section

Cross section for the resonant reaction *a* + *A* → *C** → *b* + *B* where *C** is an excited state of the compound nucleus *C*:

$$\sigma_{BW}(E) \sim \sigma_{max} \times f_R(E) \times \Gamma_a \Gamma_b$$

$$\sigma_{BW}(E) = \pi \lambdabar^2 \frac{2J_R + 1}{(2J_a + 1)(2J_A + 1)} (1 + \delta_{aA}) \frac{\Gamma_a \Gamma_b}{(E - E_R)^2 + (\Gamma/2)^2}$$

- J_R: spin of the resonance in the compound nucleus
- J_a, J_A: total angular momentum of nuclei a and A
- Spin statistical factor: $\omega = \frac{2J_R + 1}{(2J_a + 1)(2J_A + 1)} (1 + \delta_{aA})$
- Γ_a, Γ_b: partial widths for the entrance & exit channels → they are energy dependent
  - $\Gamma_i \propto P_L(E)$ → charged particles
  - $\Gamma_i \propto E^{L+1/2}$ → neutrons
  - $\Gamma_i \propto E^{2L+1}$ → γ-rays
- $\Gamma = \sum \Gamma_i$ is the total width

The Briet-Wigner formula is used for:

- Fiting data to deduce resonance properties
- Extrapolating cross section when no measurement exist
- "narrow-resonance" thermonuclear reaction rate

Subthreshold resonances

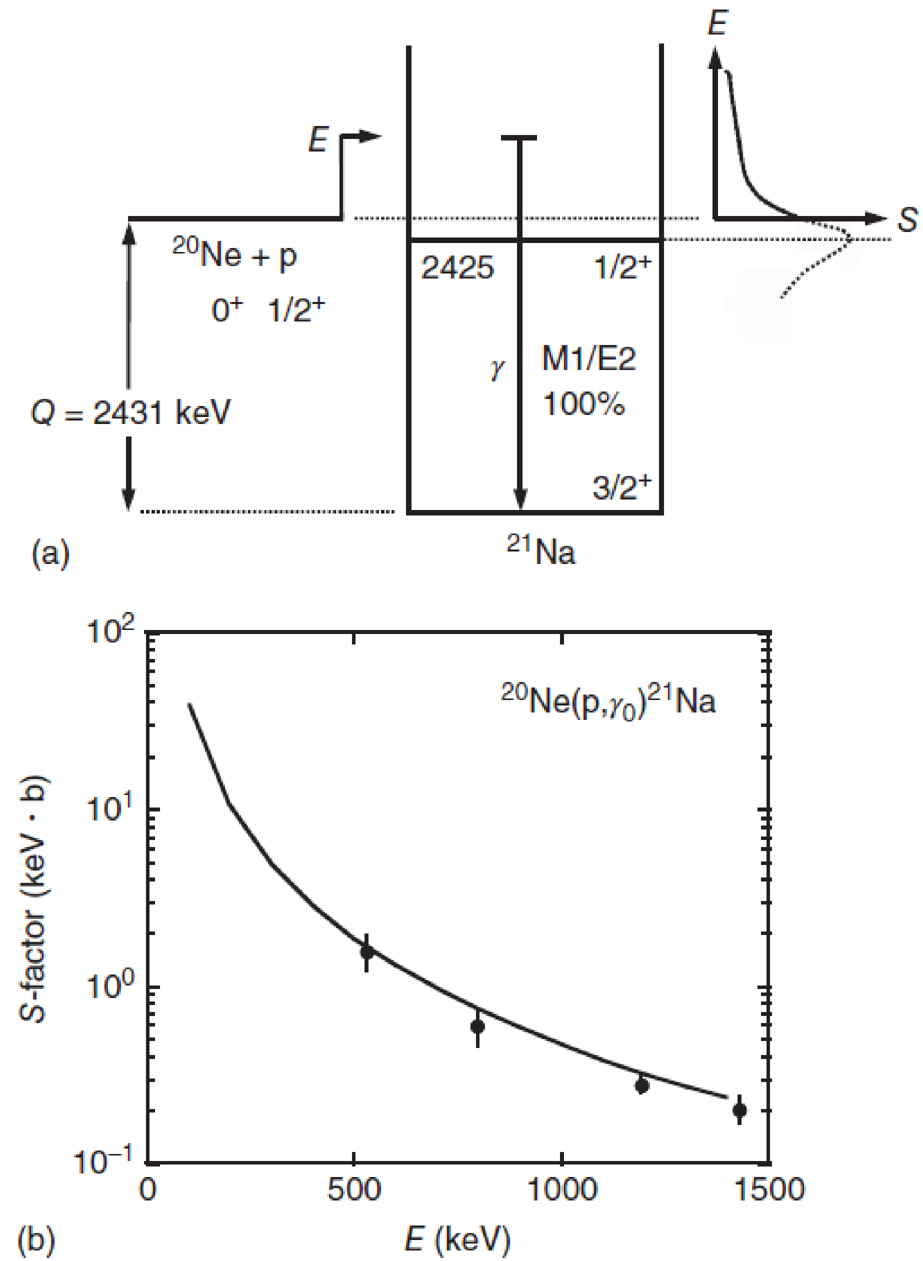
Any excited state has a finite width:

$$\Gamma = \frac{\hbar}{\tau}$$

- High-energy wing of a “bound” state can extend above the particle threshold
- *S*-factor (cross-section) can be entirely dominated by contribution of subthreshold state(s)

Example of the $^{20}Ne(p,\gamma)^{21}Na$ reaction

- E_R = 2425 - 2431 = -6 keV
- Resonance at -6 keV dominates the reaction rate



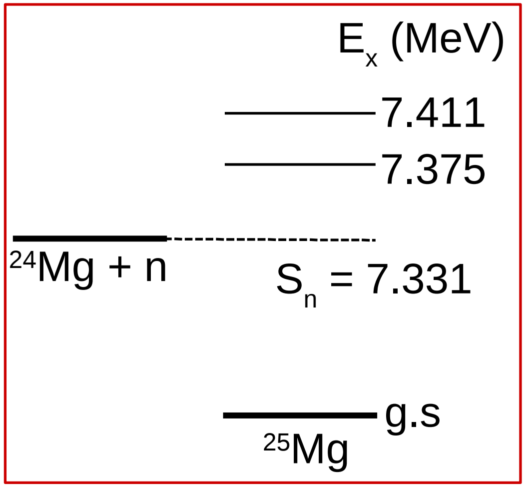
Neutron capture reactions

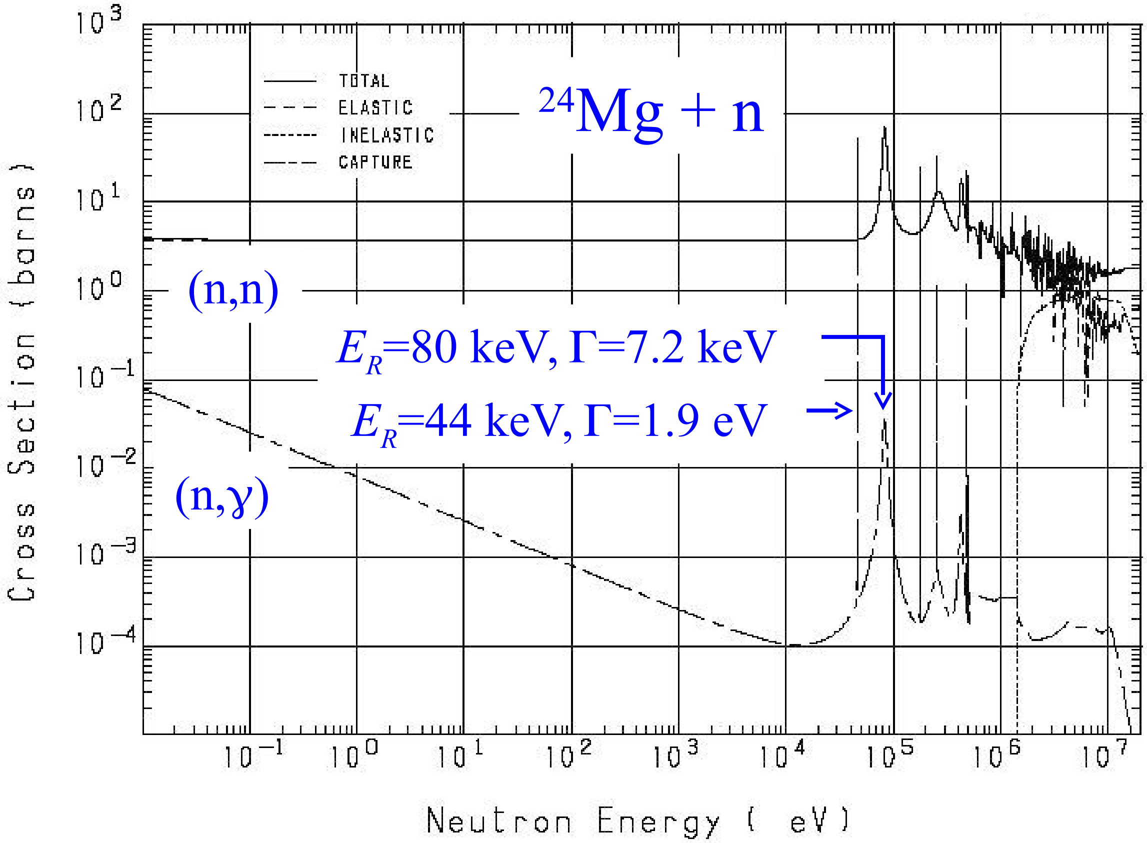
- Radiative A(n,γ)B neutron capture reaction $\sigma_{(n,\gamma)}(E) \propto \pi \lambda\!\!\!^{-2} \Gamma_n(E)\, \Gamma_\gamma(E+Q)$
- In stars $E << Q = S_n$ (neutron separation energy) → $\Gamma_\gamma(E+Q) \propto \Gamma_\gamma(Q)$
- For neutrons, $V_{coul} = 0$ ($Z_n = 0$), so only the centrifugal barrier is to be considered, the penetrability reads:

$$P_l(E) \sim E^{l+1/2}$$

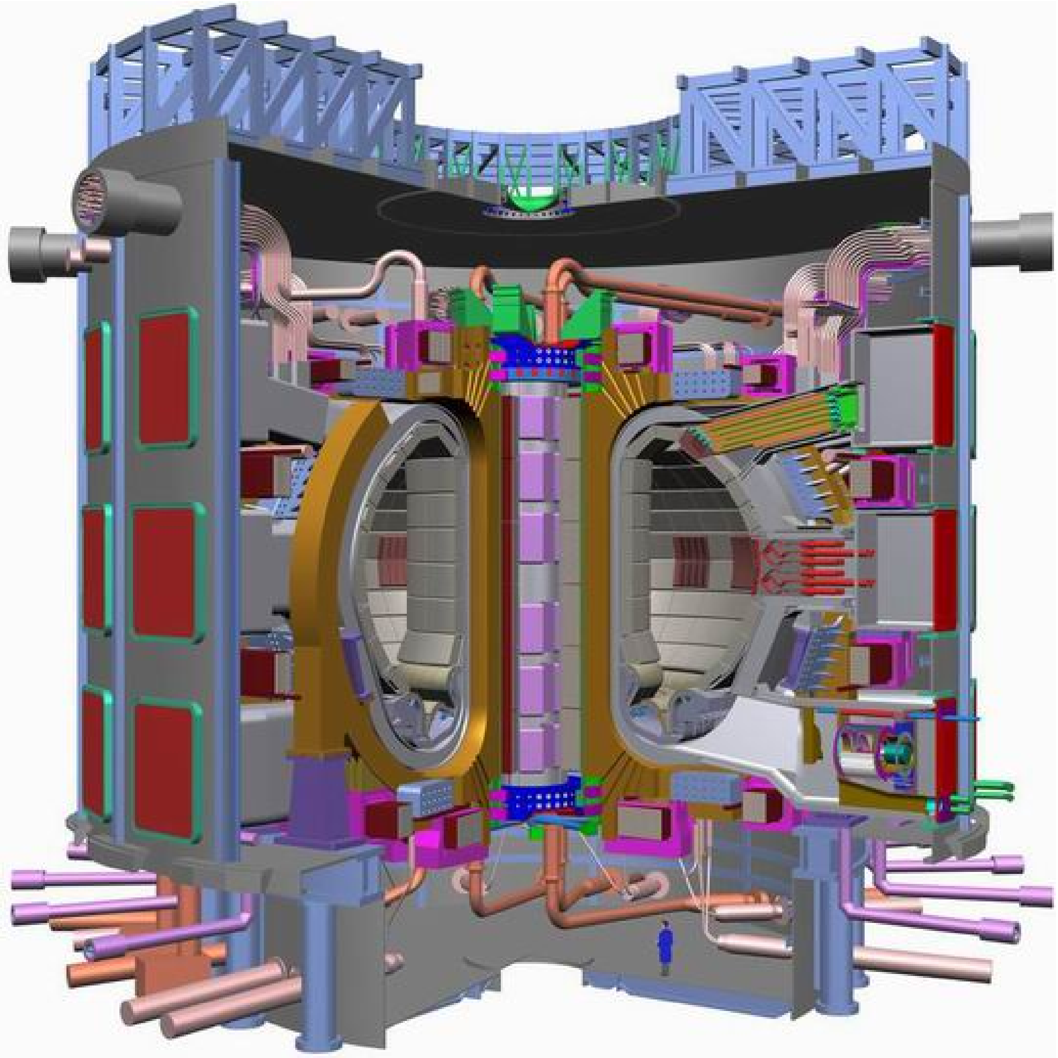
- For low-energy *s*-wave neutrons (ℓ = 0)

$$\sigma(E) \propto \frac{1}{E} E^{1/2} = \frac{1}{v}$$





2. Thermonuclear reaction rates



ITER : International Thermonuclear
Experimental Reactor (Cadarache, France)

Reaction rate

- The reaction rate is the number of reactions 1 + 2 → 3 + 4 [1(2,3)4] per unit volume and time:

$$r_{123} = \frac{dN_{12}}{dt} = \frac{N_1 N_2}{1+\delta_{12}} \int_0^\infty \sigma_{123}(v) v \phi(v) dv \equiv \frac{N_1 N_2}{1+\delta_{12}} \langle \sigma v \rangle_{123}$$

where N_i is the density of particle i (cm^{-3}), $\varphi(v)dv$ the probability for the relative speed between 1 and 2 to be in the range $[v, v+dv]$, and $\langle\sigma v\rangle_{123}$ is the reaction rate per particle pair ($cm^3\ s^{-1}$).

$1+\delta_{12} = 2$ if $1 \equiv 2$, otherwise each pair would be counted twice.

⇒ in practice $N_A\langle\sigma v\rangle$ in $cm^3\ mol^{-1}\ s^{-1}$ is tabulated in litterature

- The lifetime τ of 1 against destruction by reaction with 2 is given by:

$$\tau_2(1) = \frac{1}{\lambda_2(1)} = \left(\rho \frac{X_2}{M_2} N_A \langle \sigma v \rangle_{123}\right)^{-1}$$

ρ: mass density (g/cm^3)
X: mass fraction
M: molar mass (g/mol)
N_a: Avogadro number (at/mol)

Thermonuclear reaction rates

In a stellar plasma, the kinetic energy of nuclei is given by the thermal agitation velocity

$\Rightarrow$ thermonuclear reaction rate

For a non-degenerate perfect gas, the velocity is given by the Maxwell-Boltzmann distribution:

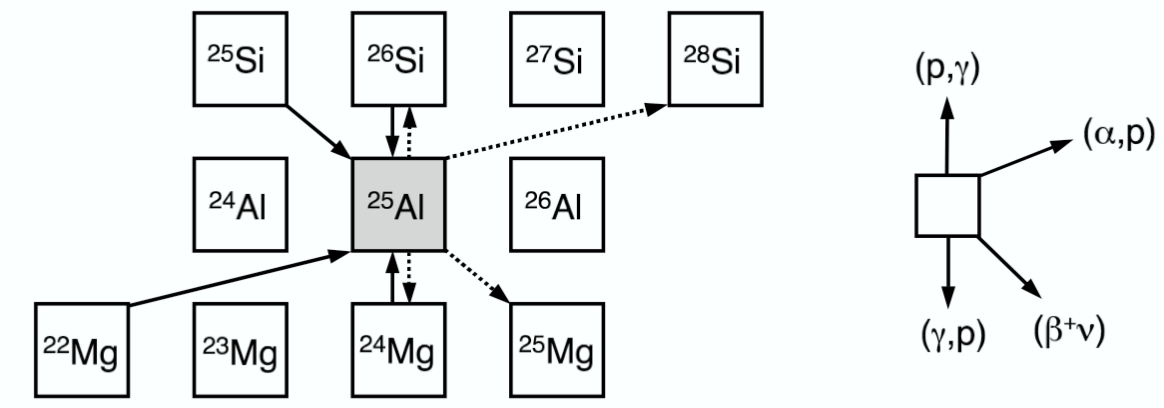
$$\phi(v)dv = \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2kT}\right) 4\pi v^2 dv$$

One obtains for the reaction rate per particle pair (in cm^3 s^{-1}) as a function of energy:

$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{123}(E) E e^{-E/kT} dE$$

The nucleosynthesis equations

- Evolution of the densities for each species:
  → system of coupled differential equations (solved numerically)
  → nuclear reaction network



$$\begin{aligned}\frac{d(N_{^{25}\mathrm{Al}})}{dt} =& N_{\mathrm{H}} N_{^{24}\mathrm{Mg}} \langle\sigma v\rangle_{^{24}\mathrm{Mg}(\mathrm{p},\gamma)} + N_{^{4}\mathrm{He}} N_{^{22}\mathrm{Mg}} \langle\sigma v\rangle_{^{22}\mathrm{Mg}(\alpha,\mathrm{p})} \\ &+ N_{^{25}\mathrm{Si}} \lambda_{^{25}\mathrm{Si}(\beta^+\nu)} + N_{^{26}\mathrm{Si}} \lambda_{^{26}\mathrm{Si}(\gamma,\mathrm{p})} + \ldots \\ &- N_{\mathrm{H}} N_{^{25}\mathrm{Al}} \langle\sigma v\rangle_{^{25}\mathrm{Al}(\mathrm{p},\gamma)} - N_{^{4}\mathrm{He}} N_{^{25}\mathrm{Al}} \langle\sigma v\rangle_{^{25}\mathrm{Al}(\alpha,\mathrm{p})} \\ &- N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\beta^+\nu)} - N_{^{25}\mathrm{Al}} \lambda_{^{25}\mathrm{Al}(\gamma,\mathrm{p})} - \ldots\end{aligned}$$

production (first two lines)

destruction (last two lines)

- Nuclear energy production rate: $\epsilon = \sum_{ijk} \frac{N_i N_j}{1+\delta_{ij}} \langle\sigma v\rangle_{ijk} Q_{ijk}$

where Q_{ijk} is the Q-value for the $i + j \rightarrow k$ reaction

Reaction rate: your turn!

In a stellar plasma, the ^{25}Al nucleus may be destroyed by the capture reaction ^{25}Al(p,γ)^{26}Si or by the β^+-decay ($T_{1/2}$ = 7.18 s). Determine the dominant destruction process among these two at a stellar temperature of T = 0.3 GK, assuming a reaction rate $N_A\langle\sigma v\rangle = 1.8\times10^{-3}$ cm^{3}mol^{-1}s^{-1}. Assume a stellar density $\rho = 10^4$ g/cm^3 and a hydrogen mass fraction $X_H = 0.7$.

Useful information: $M(^1\mathrm{H})$ = 1.0078 g/mol

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Useful information: $M(^1\mathrm{H})$ = 1.0078 g/mol

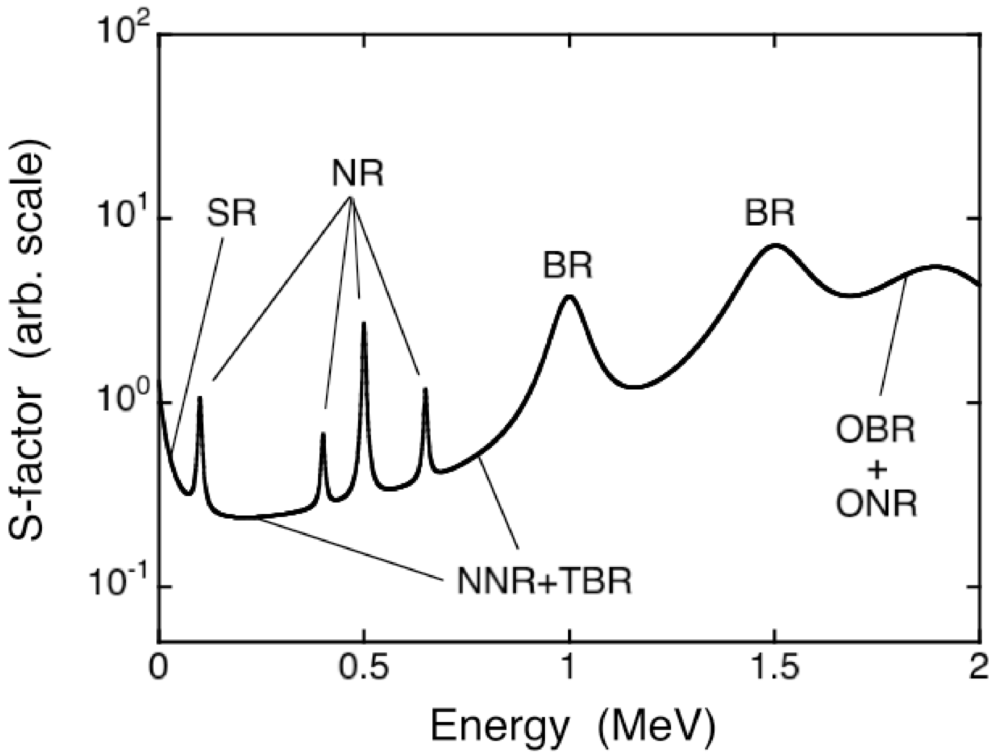
Mean lifetime of both processes:

- β^+-decay: $\tau_{\beta+}(^{25}Al) = T_{1/2}/\ln 2 = 10.36\ \mathrm{s}$
- p capture: $\tau_p(^{25}Al) = \left(\rho \frac{X_2}{M_2} N_A \langle\sigma v\rangle_{123}\right)^{-1} = \left(10^4 \times \frac{0.7}{1.0078} \times 1.8 \times 10^{-3}\right)^{-1} = 0.08\ \mathrm{s}$

$\Rightarrow$ under these conditions, the proton capture is the dominant destruction mechanism of ^{25}Al

Reaction rate calculation

Most of the time, the *S*-factor is a complex function of the energy and every nuclear reaction is a specific case



Possible contributions to the *S*-factor

- Narrow resonances (NR)
- Broad resonances (BR)
- Tail of broad resonances (TBR)
- Subthreshold resonances (SR)
- Non-resonant processes
- interferences

Thermonuclear reaction rates are calculated numerically, however several specific cases are interesting since they result in analytical expressions:

- smoothly varying *S*-factor
- narrow resonance

Gamow peak & non-resonant case

Reaction rate:

$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{123}(E) E e^{-E/kT} dE$$

If the *S*-factor is smoothly varying ("non-resonant"):

$$S(E) = \sigma(E)\, E\, e^{2\pi\eta} \cong S_0$$

$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} S_0 \int_0^\infty \boxed{e^{-2\pi\eta} e^{-E/kT}} dE$$

Gamow peak is the energy range where most reactions between 1 and 2 occur

Approximation by a Gaussian curve:

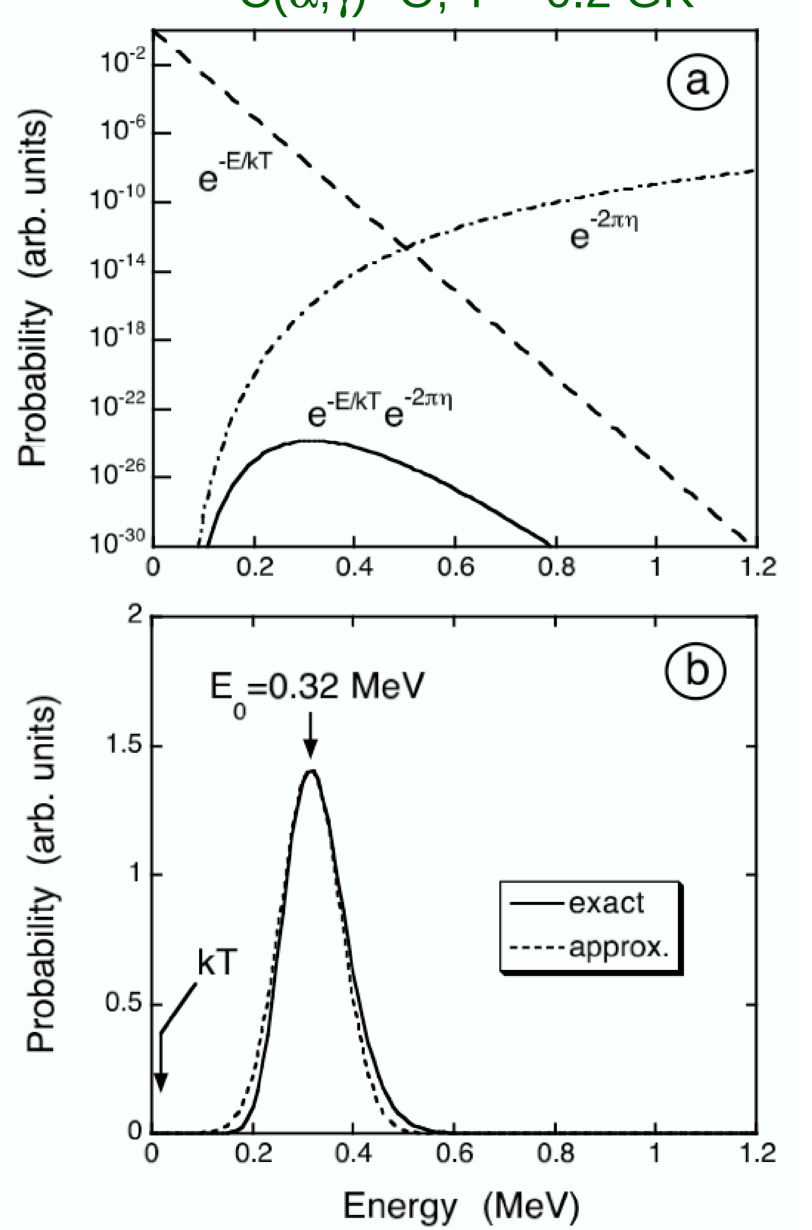
$$\exp(-2\pi\eta - E/kT) = I_{max} \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right]$$

$$E_0 = \pi k T \eta(E_0) = 1.22 \left(Z_1^2 Z_2^2 \mu_{amu} T_6^2\right)^{1/3} \quad \text{keV}$$

$$\Delta = 4\sqrt{E_0 kT/3} = 0.749 \left(Z_1^2 Z_2^2 \mu_{amu} T_6^5\right)^{1/6} \quad \text{keV}$$

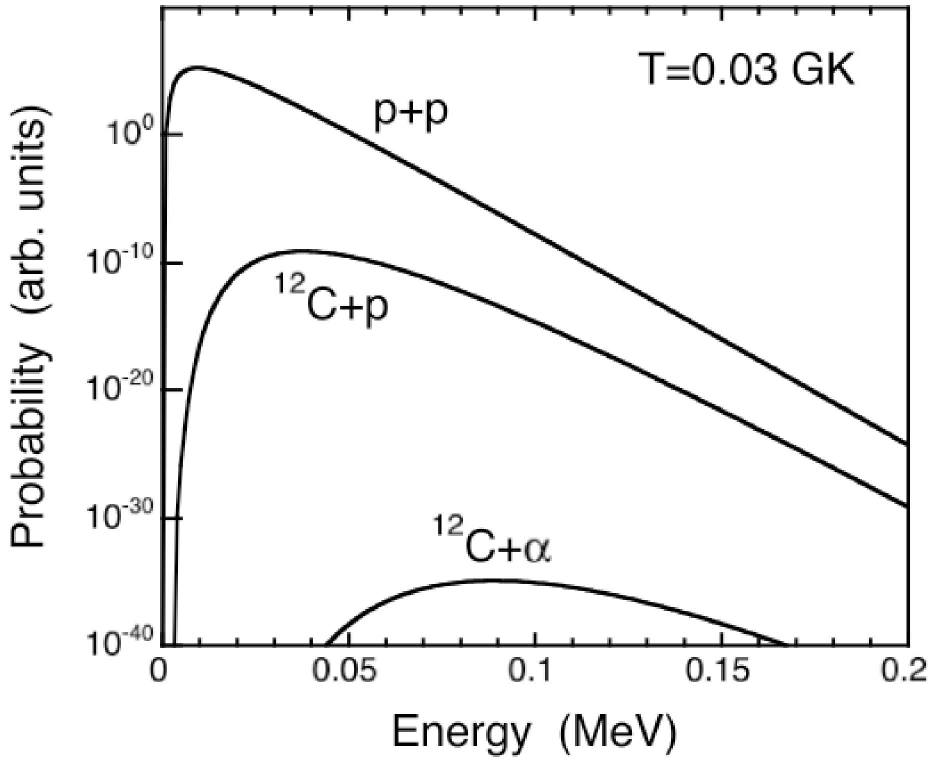
[Δ: total width at 1/e; $T_6 \equiv T$ (MK)]

$^{12}C(\alpha,\gamma)^{16}O$, T = 0.2 GK



Gamow peak properties

Gamow windows $E_0 \pm \Delta/2$



Maximum of the Gamow peak ($E = E_0$)

$$I_{max} = \exp\left(-\tau\right)$$

$$\tau = \frac{3E_0}{kT} = 42.46\left(Z_1^2 Z_2^2 \mu_{amu}/T_6\right)^{1/3}$$

$\Rightarrow$ I_{max} is strongly dependent of the product Z_1Z_2

Important properties

- Gamow peak shift to higher energy for increasing charges Z_1, Z_2
- Area under Gamow peak decreases drastically with increasing charges Z_1 and Z_2

reaction	Coulomb barrier (keV)	E_0 (keV)	Δ (keV)	Area Gamow peak ($I_{max}\Delta$)
p+p	554	9.4	11.4	2.2×10^{-4}
^{12}C+p	2020	38.0	22.9	1.9×10^{-18}
^{12}C+α	3429	89.1	35	4.8×10^{-44}

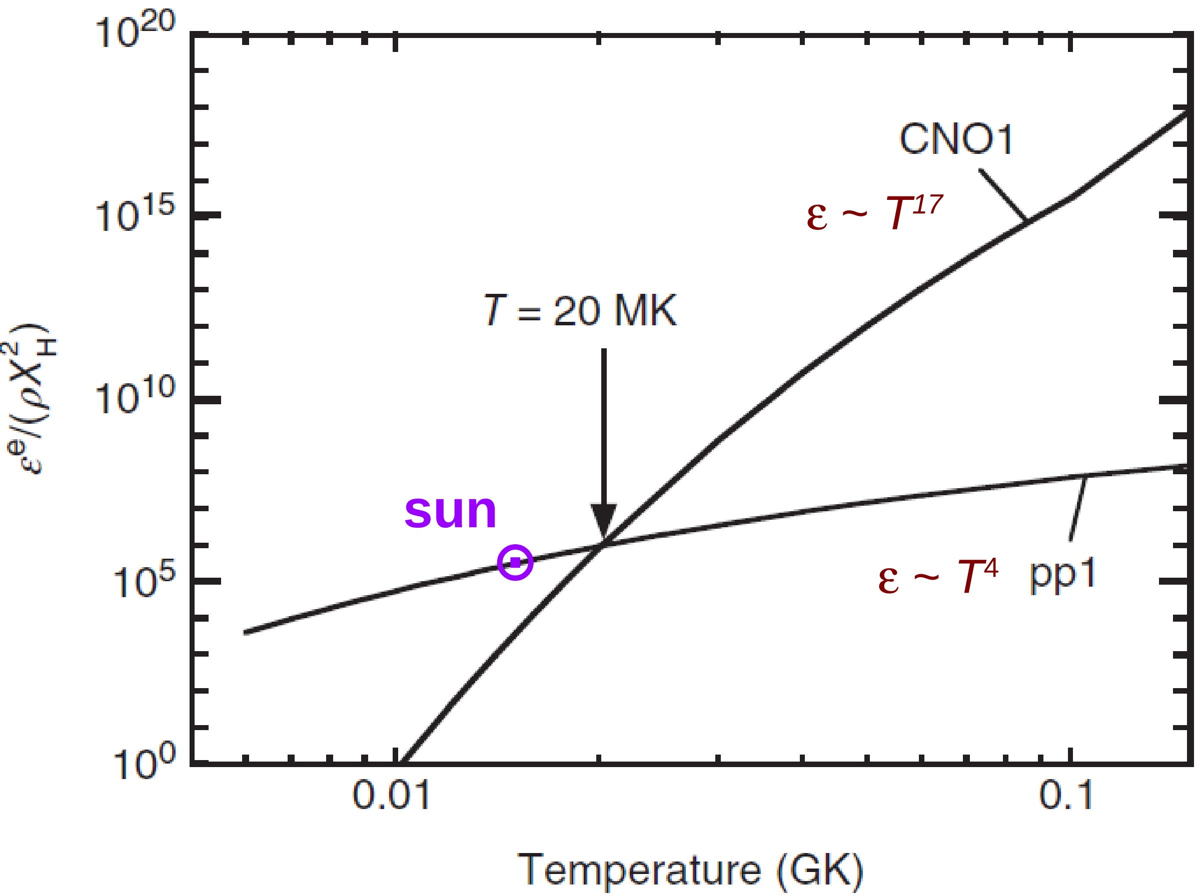
Reactions with the smallest Coulomb barrier produce most of the energy and are consumed rapidly
→ successive burning stages

Non-resonant reaction rates

Reaction rate: $\langle\sigma v\rangle_{123} = \sqrt{\frac{8}{\mu\pi}}\frac{1}{(kT)^{3/2}}S(0)\sqrt{\pi/2}\; I_{max}\Delta$

with $S(E_0)$ in keV b: $\langle\sigma v\rangle_{123} = 4.33\times10^5\,\frac{\tau^2\exp(-\tau)}{Z_1 Z_2 \mu_{amu}}\,S(E_0)\ \mathrm{cm^3 mol^{-1} s^{-1}}$

Energy production rate



Temperature dependence

$$\langle\sigma v\rangle_{123} \propto T^{(\tau-2)/3}$$

In our Sun (now), $T_6 \approx 16$

- $\langle\sigma v\rangle_{p+p} \propto T^{3.9}$
- $\langle\sigma v\rangle_{^{12}\mathrm{C}+p} \propto T^{17.8}$

Gamow window: your turn!

The $^{15}O(\alpha,\gamma)^{19}Ne$ capture is one of the hot-CNO break-out reaction occuring in X-ray bursts at about 0.4 GK.

1) Calculate the Gamow peak energy and width in these conditions.
2) Calculate the corrresponding excited energy range in the compound nucleus.
3) What are the relevant ^{19}Ne states for this reaction in these conditions? Use the nndc ressource (https://www.nndc.bnl.gov/ensdf/)
4) What is the most likely contributing state to the $^{15}O(\alpha,\gamma)^{19}Ne$ reaction rate? Hint: find the state corresponding to the lowest orbital angular momentum

Useful information: $J^{\pi}(^{15}O) = 1/2^{-}$, $m(^{15}O) = 15.0031$ u, $m(^{4}He) = 4.0026$ u, $m(^{19}Ne) = 19.0019$ u, u = 931.4 MeV/c^2

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Solutions:

1) E_0 = 617 keV; Δ = 337 keV
2) $Q = S_\alpha = m(^{15}O)c^2 + m(^{4}he)c^2 - m(^{19}Ne)c^2$ = 3.539 MeV

→ excitation energy range between $E_{x,inf} = S_\alpha + E_0 - \Delta/2$ = 3978 keV

$E_{x,sup} = S_\alpha + E_0 + \Delta/2$ = 4315 keV

3) $E_x(^{19}Ne)$ = 4033-, 4140-, and 4197-keV
4) Entrance channel spin $s = 1/2$, $\pi = -1$;

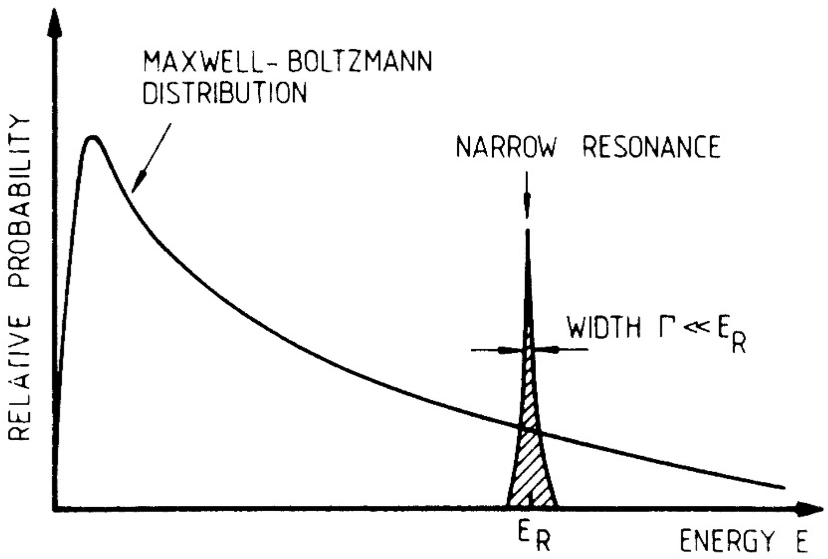
→ $E_x(^{19}Ne)$ = 4033 keV ($3/2^+$) ℓ=1; 4140 keV ($9/2^-$) ℓ=4; 4197 keV ($7/2^-$) ℓ=4

The narrow resonance case (1)

- Contribution to the reaction rate of a resonance at the energy E_R close to E_0:

$$\langle\sigma v\rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{BW}(E) E e^{-E/kT} dE$$

with $$\sigma_{BW}(E) = \pi\lambda\!\!\!^{-2}\,\omega \frac{\Gamma_a \Gamma_b}{(E-E_R)^2 + (\Gamma/2)^2}$$



- For a narrow resonance: Maxwell-boltzmann distribution ~ constant

$$\langle\sigma v\rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{E_R e^{-E_R/kT}}{(kT)^{3/2}} \int_0^\infty \sigma_{BW}(E) dE$$

- If the partial widths (Γ_i) are constants over $\Gamma << E_R$: $$\int_0^\infty \sigma_{BW}(E) dE = 2\pi^2 \lambda\!\!\!^{-2}_R \omega \frac{\Gamma_a \Gamma_b}{\Gamma}$$

$$\langle\sigma v\rangle_{123} = \left(\frac{2\pi}{\mu kT}\right)^{3/2} \hbar^2\, \omega\gamma\, e^{-E_R/kT} \qquad \omega\gamma = \omega \frac{\Gamma_a \Gamma_b}{\Gamma}$$ is the resonance strength

The narrow resonance case (2)

Contribution of a single narrow resonance to the stellar thermonuclear reaction rate:

$$N_A \langle \sigma v \rangle = 1.54 \times 10^{11} \left(A T_9\right)^{-3/2} \, \omega\gamma \, \exp\left(-11.605 \frac{E_R}{T_9}\right) \, \mathrm{cm^3 \, mol^{-1} \, s^{-1}}$$

with $\omega\gamma$, E_R in MeV

- Resonance energy E_R
  - Strong energy dependence (in exponential term!)
    → few keV uncertainties in resonance energy implies large uncertainties on reaction rate
    → e.g. ΔE_R = 6 keV ⇒ factor of 2 on the reaction rate!
  - $E_R = E_X - Q$ → Accurate excitation energies and masses are needed!

- Resonance strength $\omega\gamma$
  - Depends mainly on the total (Γ) and partial widths (Γ_i)

$$\omega\gamma = \frac{2J_R + 1}{(2J_a + 1)(2J_A + 1)} \frac{\Gamma_a \Gamma_b}{\Gamma}$$

  - Consider a resonant state with only two open channels: $\Gamma = \Gamma_a + \Gamma_b$
    - If $\Gamma_a << \Gamma_b$, then $\Gamma \approx \Gamma_b$ ⇒ $\omega\gamma \approx \omega\Gamma_a$
    - If $\Gamma_b << \Gamma_\alpha$, then $\Gamma \approx \Gamma_a$ ⇒ $\omega\gamma \approx \omega\Gamma_b$

The reaction rate is determined by the smallest partial width

Resonant case: your turn!

The $^{13}N(\alpha,p)^{16}O$ reaction plays an important role in explosive He burning in massive stars at about 0.6 GK.

1) Calculate the Gamow peak energy and width in these conditions.
2) What is compound nucleus? Calculate the excited energy range of interest.
3) What are the relevant states for this reaction in these conditions? Say whether resonant states are narrow or broad (see https://www.nndc.bnl.gov/ensdf/).
4) Explain why these resonant states decay mainly by proton emission.
5) Write the resonance strength for the narrow states; what is the important parameter to be measured experimentally?

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4) Explain why these resonant states decay mainly by proton emission.
5) Write the resonance strength for the narrow states; what is the important parameter to be measured experimentally?

Solutions:

1) E_0 = 732 keV; Δ = 449 keV

2) $E_{x,inf} = S_\alpha + E_0 - \Delta/2$ = 6.327 MeV

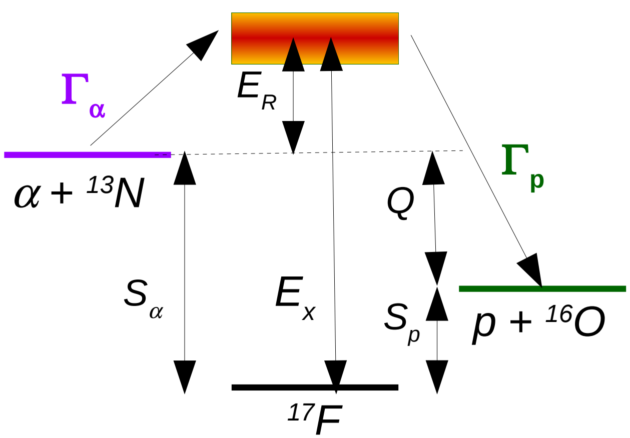
$E_{x,sup} = S_\alpha + E_0 + \Delta/2$ = 6.776 keV

3) $E_x(^{17}F)$ = 6560 keV (BR), 6697 keV (NR)

4) $S_\alpha >> S_p \rightarrow P_{\alpha+13N}(E_R) << P_{p+16O}(E_R+Q) \rightarrow \Gamma_\alpha << \Gamma_p$

5) $\omega\gamma = \omega\Gamma_\alpha\Gamma_p/\Gamma$ with $\Gamma = \Gamma_\alpha + \Gamma_b$. Since $\Gamma_\alpha << \Gamma_p$, $\omega\gamma \approx \omega\Gamma_\alpha$

→ α-particle partial width (Γ_α) should be a prime objective for an experimental study



The general resonance case

- In the most general case, the Breit-Wigner formula with energy-dependent partial widths should be used

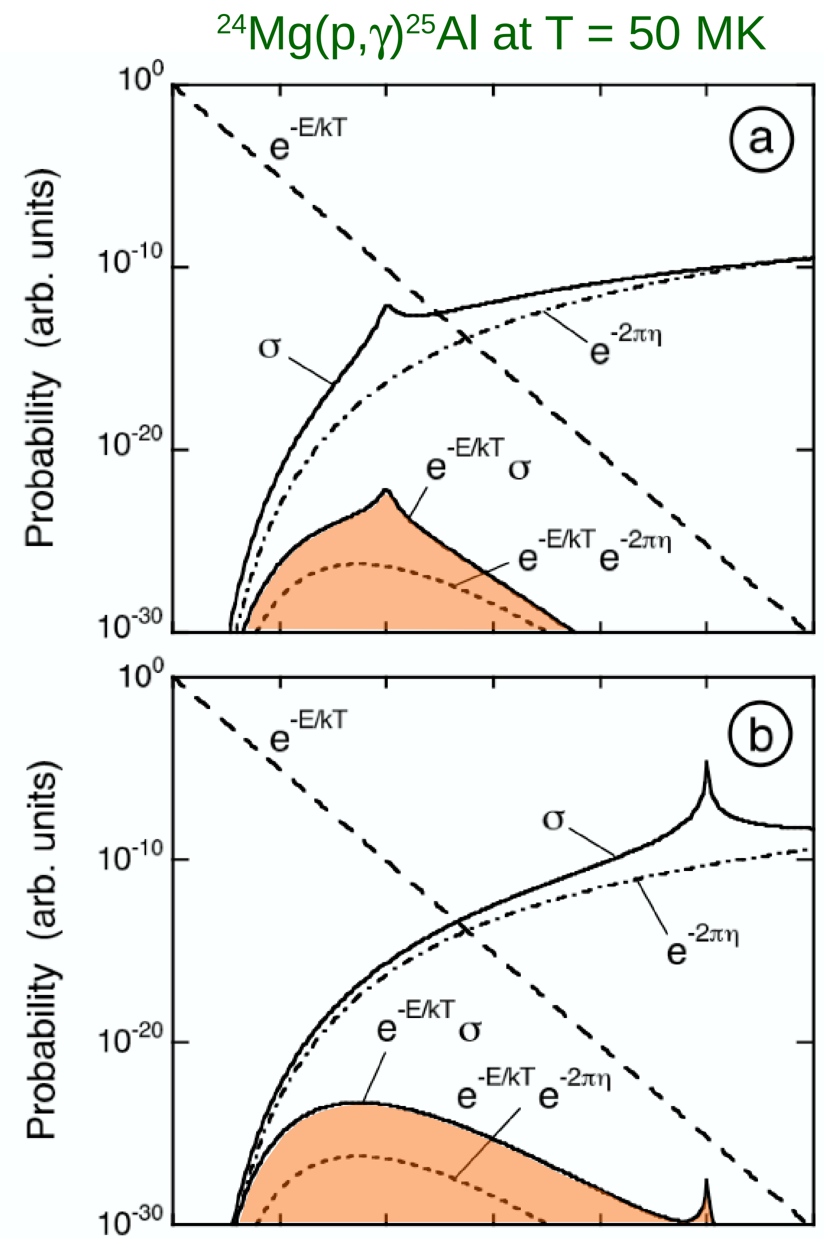
$$\langle\sigma v\rangle_{123} = \sqrt{\frac{8}{\mu\pi}}\frac{1}{(kT)^{3/2}}\int_0^\infty \sigma_{BW}(E)\,E e^{-E/kT}dE$$

with

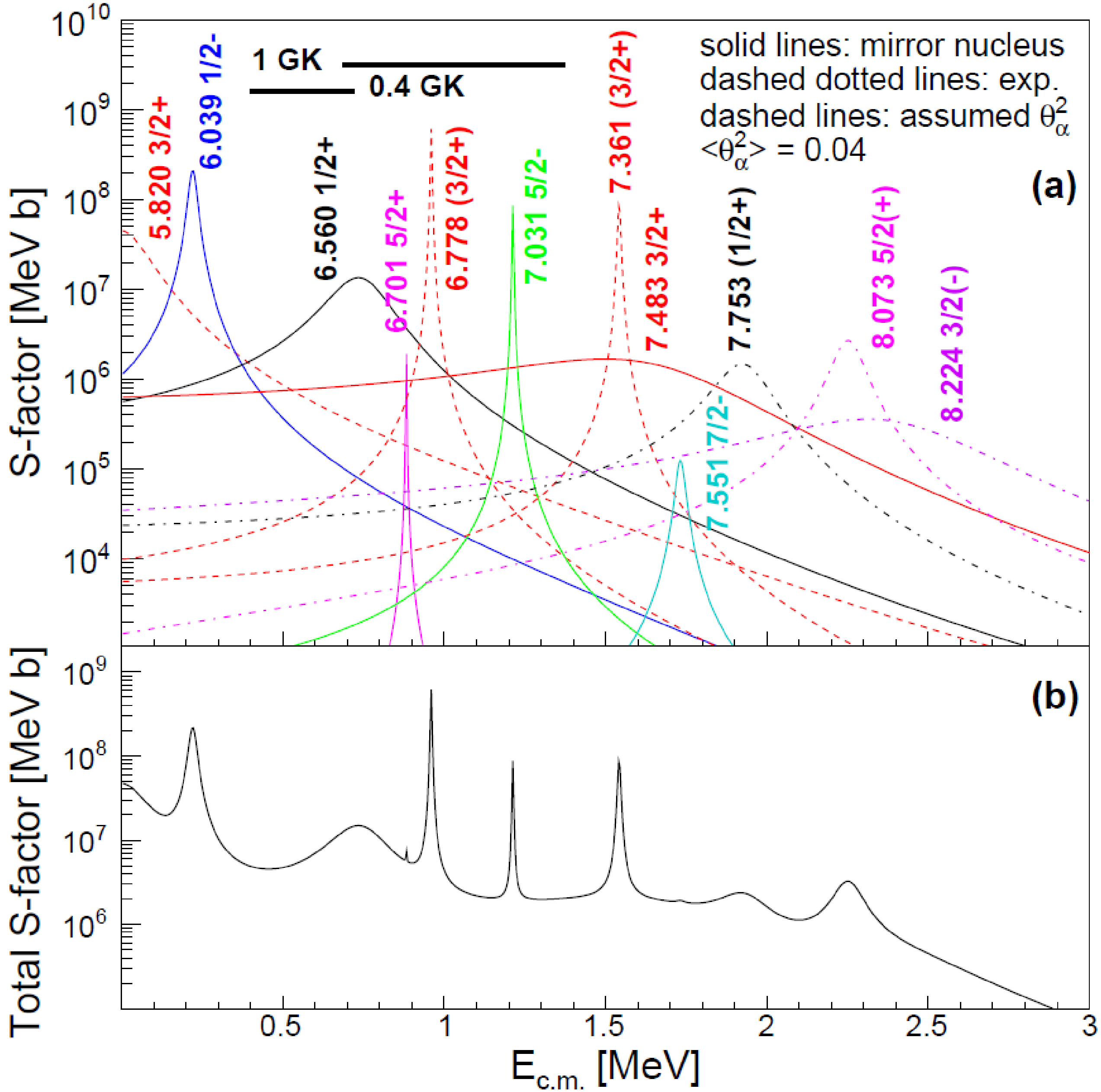
$$\sigma_{BW}(E) = \pi\lambda\!\!\!^{-2}\,\omega\,\frac{\Gamma_a(E)\Gamma_b(E+Q)}{(E-E_R)^2+(\Gamma(E)/2)^2}$$

$\Rightarrow$ numerical integration

- When the resonance is outside the Gamow peak
  - Contribution to the reaction rate through its tail
  - *S*-factor of resonance tail is slowly varying with energy
    → similar treatment as for the Direct Capture process



A typical case: $^{13}N(\alpha,p)^{16}O$



Meyer+ PRC (2021)

Direct and resonant capture: $^{32}Cl(p,\gamma)^{33}Ar$

Spectroscopic information

TABLE V. Nonresonant direct capture transitions and the astrophysical S factors; resonance energies, γ widths, proton widths, and resonance strengths for $^{32}Cl(p,\gamma)^{33}Ar$.

$^{32}Cl(p,\gamma)^{33}Ar$ $\quad Q = 3.34$ MeV

direct

E_x	J^π	ℓ_i	$n\ell_f$	C^2S_f	$S(E_0)$ (MeV b)
0.00	$\frac{1}{2}^+_1$	p	$2s_{1/2}$	0.080	7.00×10^{-3}
		p	$1d_{3/2}$	0.672	6.14×10^{-3}
1.34	$\frac{3}{2}^+_1$	p	$1d_{3/2}$	0.185	2.62×10^{-3}
1.79	$\frac{5}{2}^+_1$	p	$1d_{3/2}$	0.145	2.74×10^{-3}
2.47	$\frac{3}{2}^+_2$	p	$2s_{1/2}$	0.031	6.16×10^{-3}
		p	$1d_{3/2}$	0.167	1.67×10^{-3}
3.15	$\frac{3}{2}^+_3$	p	$2s_{1/2}$	0.068	1.46×10^{-2}
		p	$1d_{3/2}$	0.516	3.01×10^{-3}

S_p = 3.34 MeV

resonant

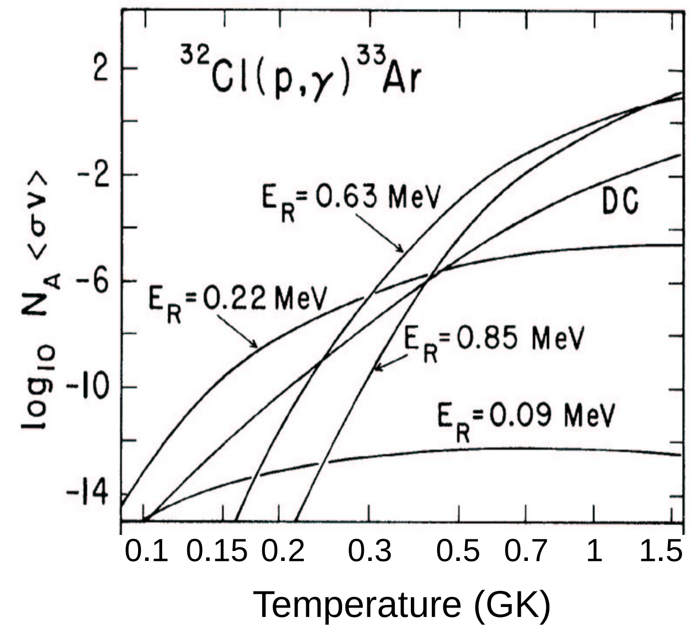
E_x	E_p	J^π	Γ_γ (eV)	Γ_p (eV)	$\omega\gamma$ (eV)
3.43	0.09	$\frac{5}{2}^+_2$	1.77×10^{-2}	8.7×10^{-18}	8.7×10^{-18}
3.56	0.22	$\frac{7}{2}^+_2$	1.94×10^{-3}	1.13×10^{-9}	1.51×10^{-9}
3.97	0.63	$\frac{5}{2}^+_3$	1.54×10^{-2}	2.22×10^{-2}	9.09×10^{-3}
4.19	0.85	$\frac{1}{2}^+_2$	1.54×10^{-1}	46.74	5.12×10^{-2}
4.73	1.39	$\frac{3}{2}^+_4$	8.48×10^{-2}	100.3	5.65×10^{-2}

Weak energy dependence of γ-ray width

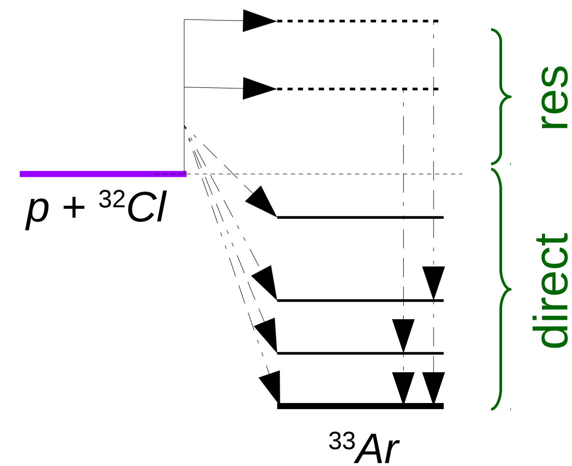
Strong energy dependence of proton width

Resonance strength

Reaction rate



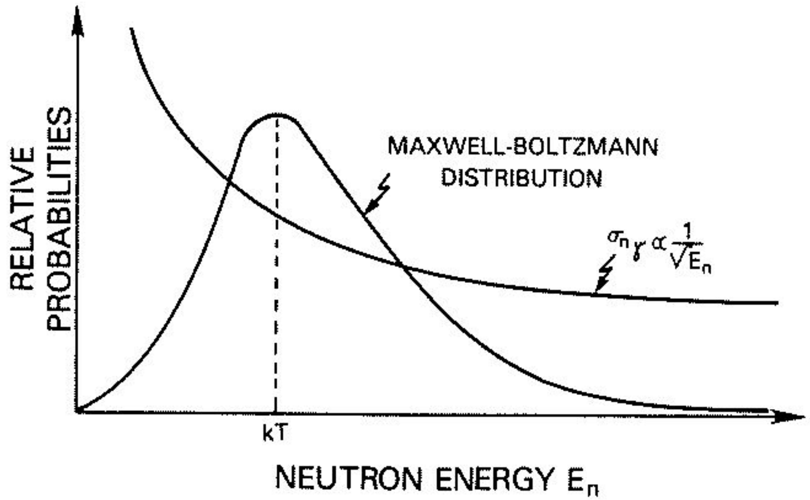
Herndl, PRC52, 78 (1995)



Contribution of resonances vary as a function of temperature

Neutron capture reaction rates

- Neutrons in stars are quickly thermalized
  → *kT* is the most probably capture energy



- Non-resonant component
  - For *s*-wave (ℓ = 0) neutron capture

$$\sigma(v) = \frac{K}{v} \qquad K \text{ is a constant}$$

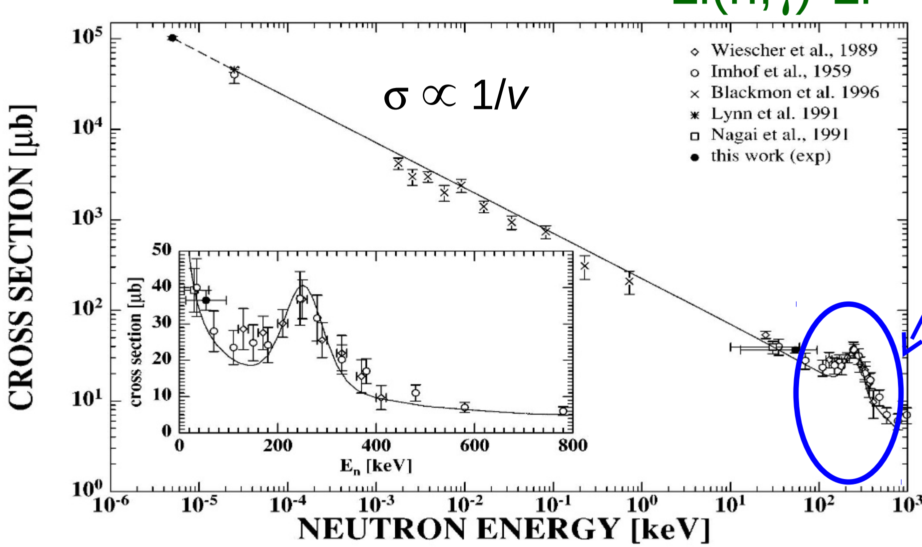
  - Reaction rate:

$$\langle \sigma v \rangle_{(n,\gamma)} = \int_0^\infty \sigma_{(n,\gamma)}(v) v \phi(v) dv = K \int_0^\infty \phi(v) dv = K$$

    → constant reaction rate!
    → Independent of temperature

- Resonant component
    → Breit-Wigner treatment

- Cross section can be measured directly



Numerical calculation of reaction rates

- Ingredients for calculating reaction rates
  - Resonance energy
  - Resonance strength
  - *S*-factor
  - Partial widths
  - …

- It's easy to compute a reaction rate…. → nominal reaction rate

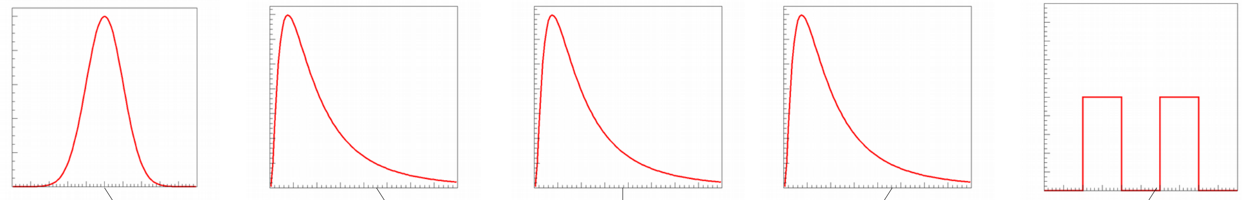
$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{123}(E) E e^{-E/kT} dE$$

- … but what about uncertainties?
  - Interferences
  - Spin/parity
  - Relation between resonance energy and partial widths

how do define "upper" / "lower" reaction rates?

Monte-Carlo approach

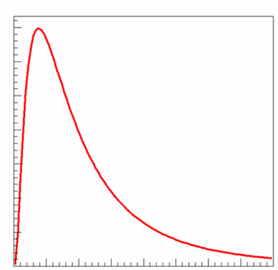
Resonance energy | Resonance strength | *S*-factor | Partial with | Interferences



Experimental nuclear physics input

formalism

$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma_{123}(E) E e^{-E/kT} dE$$

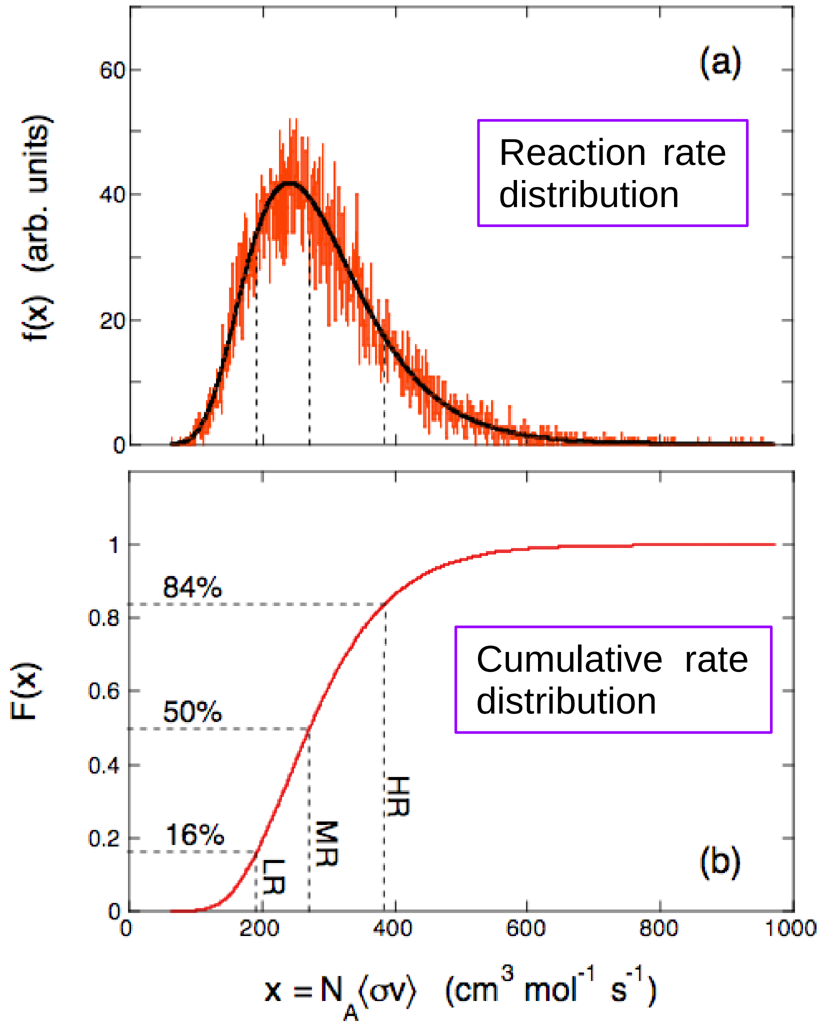


Reaction rate output

Log-normal density probability function:

$$f(x>0) = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

Low, recommended and high reaction rates



Schematic example

- ^{20}Ne(α,γ)^{26}Mg at 500 MK
- E_R = 300 ± 15 keV
- ωγ = 4.1 ± 0.2 eV
- 10000 samples

- Definition of statistically meaningful thermonuclear reaction rates
  - Cumulative distribution function

$$F(x) = \int_0^x f(x)dx$$

  - Low, recommended, high reaction rates → 16th, 50th, 84th percentile of the cumulative rate distribution

RateMC code + evaluation of reactions involving targets in *A*=14-40 mass region

Iliadis+ NPA841, 31 (2010)

Additional effects in stellar environment

- In extreme stellar environments additional effects (other than temperature and density) affect the thermonuclear reaction rates
- In particular, experimental laboratory reaction rates need to be corrected (theoretically) to obtain stellar reaction rates

  → two main effects to consider

1) Thermally excited target

For high temperatures photons can excite the nuclei. Reactions on excited target nuclei can have different angular momentum and parity selection rules and have a somewhat different *Q*-value.

2) Electron screening

Atoms are fully ionized in a stellar environment, but the electron gas shields the nuclei and affects the effective Coulomb barrier.

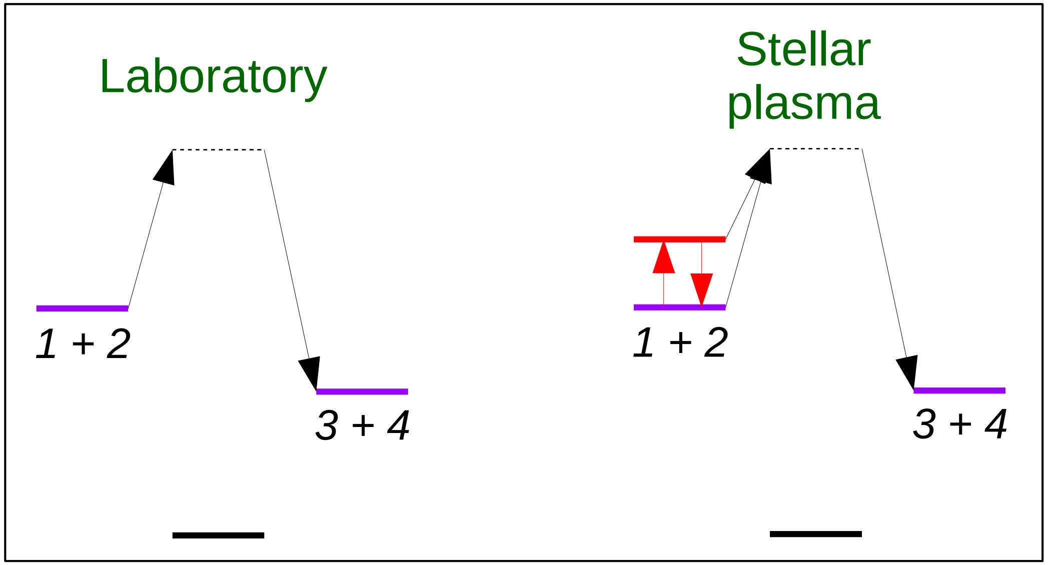
Reactions measured in the laboratory are also screened by the atomic electrons, but the screening effect is different

Thermally excited target nuclei

- At elevated stellar temperatures, the nuclei will be thermally excited

  → photoexcitation, inelastic scattering…

$$\frac{N_{ex}}{N_{gs}} = \frac{2J_{ex}+1}{2J_{gs}+1} e^{-E_{ex}/kT}$$



- The Stellar Enhancement Factor (SEF) is the ratio of stellar to laboratory reaction rates:

$$SEF \equiv \frac{N_A \langle \sigma v \rangle^*_{123}}{N_A \langle \sigma v \rangle_{123}}$$

→ must be calculated theoretically

  - Usually only a very small correction (SEF ~ 1) because kT ~ 1 – 100 keV smaller than the level spacing at low energies (~ MeV)
  - But should be considered (*i*) at high temperatures, (*ii*) when a low lying excited state exist in the target nuclei, (*iii*) when populated state has very different reaction rate (because of different spin, parity...)

    → example of ^{26}Al isomeric state ($T_{1/2}$ = 6.34 s) at E_x = 228 keV

Electron screening

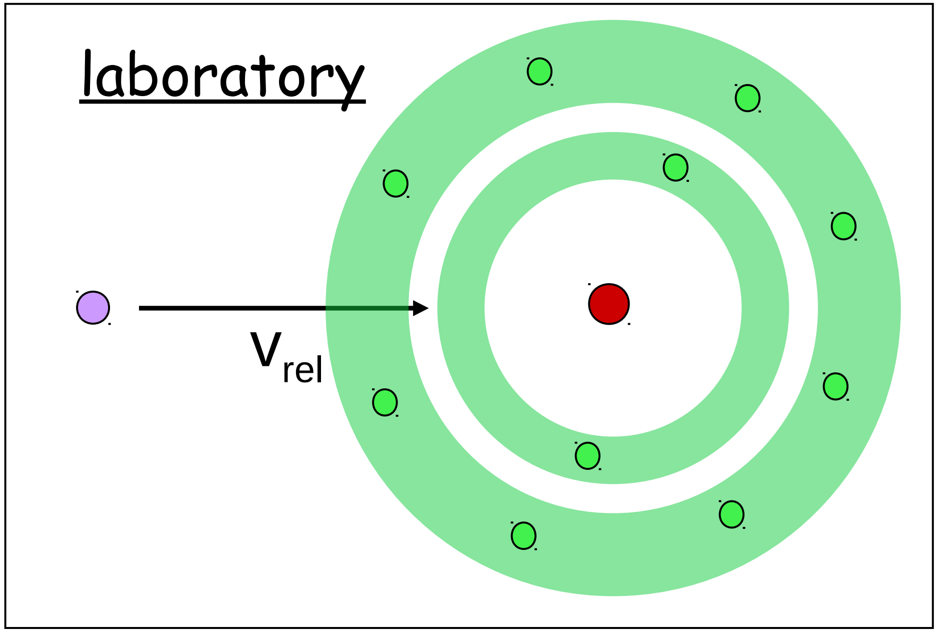
- **In the laboratory**, reaction between a charged projectile and a neutral atom (in general)

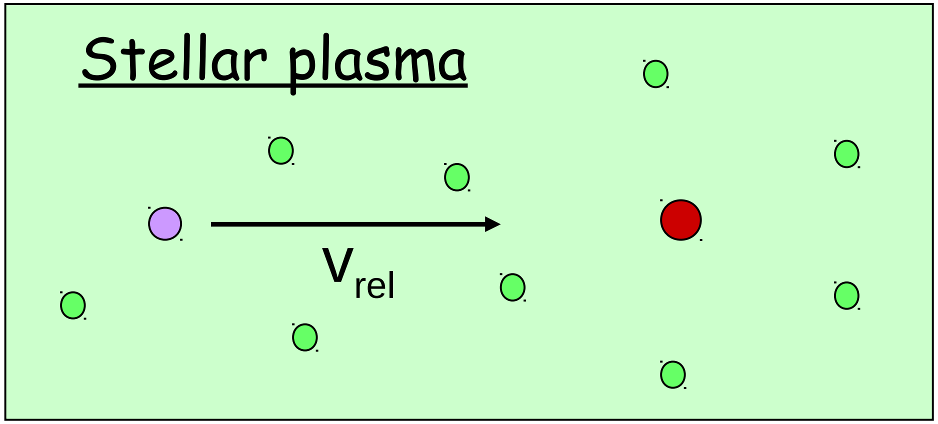
  → electron screening of the Coulomb potential from the target nucleus

- **In stars**, atoms are ionized within an electron plasma

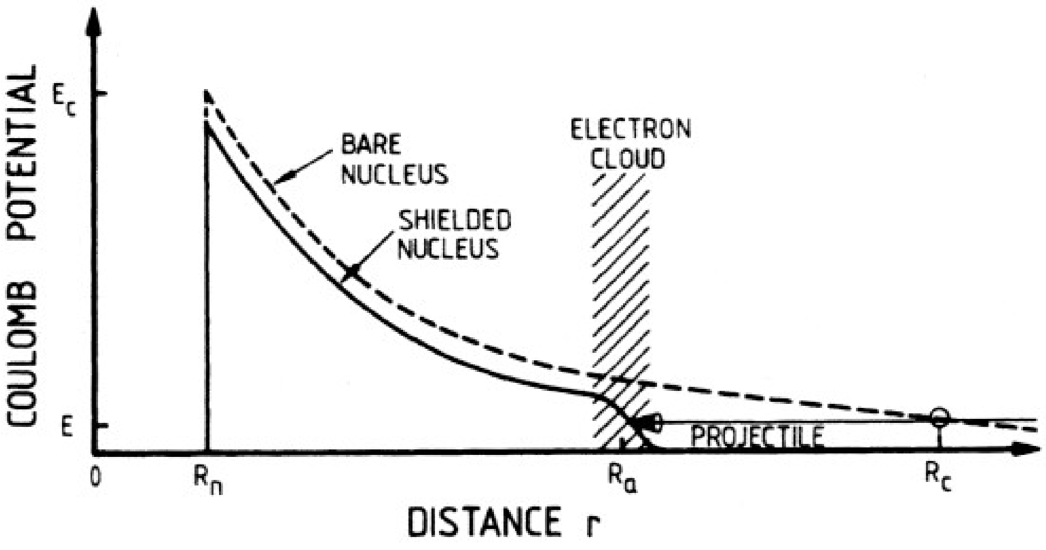
  → screening by the plasma electrons

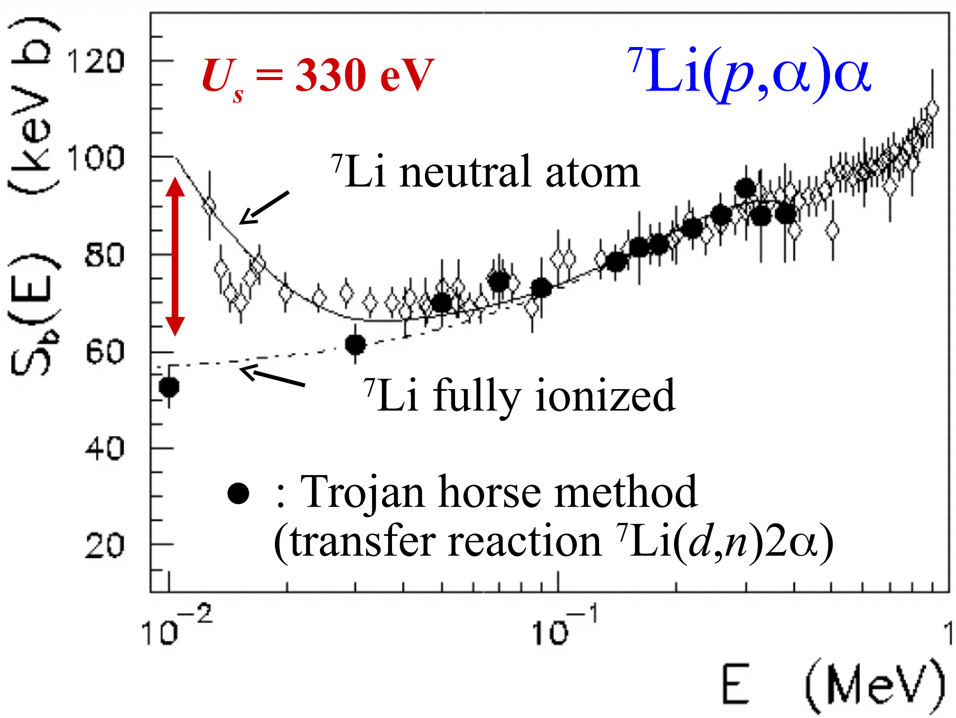
- Strategy:
  - Estimate the cross section for the reaction between fully ionized nuclei (bare cross section σ_b)
  - Deduce the stellar cross section, reaction rate from correction, which depends on stellar plasma conditions (ρ and T)





Electron screening in the laboratory





- Incident particle feels the following potential:

$$V = \boxed{\frac{Z_1 Z_2 e^2}{r}} + \boxed{U_s}$$

Coulomb potential — screening potential

- Screening potential is attractive ($U_s < 0$)

$$U_s = -\frac{Z_1 Z_2 e^2}{R_a}$$

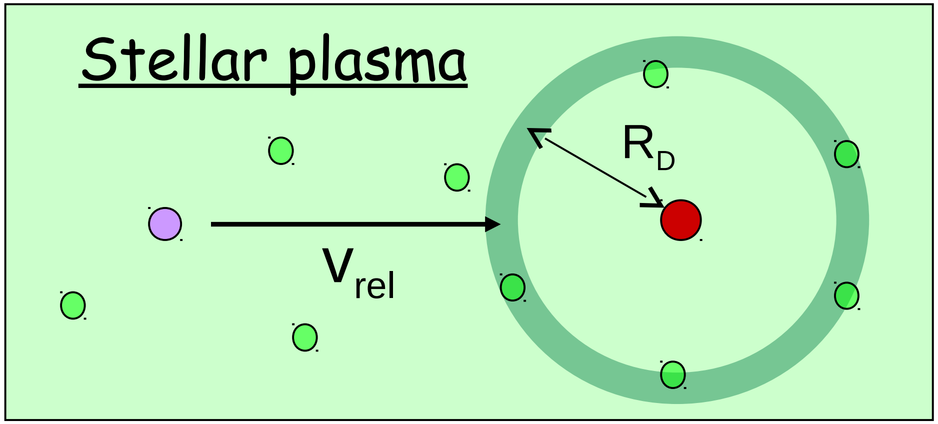
R_a is the “characteristic” atomic radius

- Enhancement of the cross section for the neutral atom

$$\frac{\sigma_s}{\sigma_b} \approx \exp\left(-\pi\eta U_s/E\right)$$

Electron screening in stars

- In stellar cores, ions are fully ionized and surrounded by electrons
- In an almost perfect gas, the characteristic distance from the free electron cloud to the ion is the Debye-Hückel radius R_D



- Corresponding screening potential: $U_s = -\frac{Z_1 Z_2 e^2}{R_D}$

- Shielded reaction rate:

$$\langle \sigma v \rangle_{123} = \sqrt{\frac{8}{\mu\pi}} \frac{1}{(kT)^{3/2}} \int_0^\infty S_{123}(E)\, e^{-2\pi\eta} \boxed{e^{-\pi\eta U_s/E}} e^{-E/kT} dE$$

- Correction factor *f*:

$$\langle \sigma v \rangle_{screened} = f_s \langle \sigma v \rangle_{bare} \quad \text{with} \quad f_s = \exp\left(-\pi\eta(E_0) U_s / E_0\right) = \boxed{\exp\left(-\frac{U_s}{kT}\right)}$$

(E_0 is the energy of the Gamow peak)

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